Cubic Forms in Thirty-Two Variables

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Phil. Trans. R. Soc. Lond. A 1959 **251**, 193-232
doi: 10.1098/rsta.1959.0002

CUBIC FORMS IN THIRTY-TWO VARIABLES

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(*Received* 10 *December* 1957)

CONTENTS

It is proved that if $C(x_1, \ldots, x_n)$ is any cubic form in n variables, with integral coefficients, then the equation $C(x_1, \ldots, x_n) = 0$ has a solution in integers $x_1, \ldots, x_n$, not all 0, provided n is at least 32. The proof is based on the Hardy–Littlewood method, involving the dissection into parts of a definite integral, but new principles are needed for estimating an exponential sum containing a general cubic form. The estimates obtained here are conditional on the form not splitting in a particular manner; when it does so split, the same treatment is applied to the new form, and ultimately the proof is made to depend on known results.

1. Introduction

A classical theorem, due to Meyer, states that if $n \geqslant 5$ every indefinite quadratic form $Q(x_1, \ldots, x_n)$ in n variables with integral coefficients represents zero, that is, the equation $Q = 0$ has a solution in integers $x_1, \ldots, x_n$, not all 0. It has long been conjectured that some similar result holds for forms of higher degree, but until recently such results had been proved only for special forms, such as $c_1 x_1^k + \ldots + c_n x_n^k$, which are amenable to treatment by the methods devised for Waring's Problem.

It has recently been proved independently by D. J. Lewis, by myself, and by B. J. Birch (in order of priority) that there is a number N such that every cubic form in n variables, with integral coefficients, is a zero form if $n \geqslant N$. The work of Lewis (1957*b*) is based on a theorem of Brauer (1945). The work of Birch (1957) uses similar ideas but goes further; it proves the analogous result for all forms of odd degree, with a value of N depending on the degree, as indeed it must. In both treatments the problem for the general cubic form is ultimately reduced to the corresponding problem for a form of the type $c_1 x_1^3 + \ldots + c_m x_m^3$, for which the result is known to hold.

My own proof, presented here, has the merit of giving a reasonable value for N, namely 32. Stated formally, the result is:

Theorem. *Let*

$$C(x_1, \ldots, x_n) = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} c_{ijk} x_i x_j x_k \tag{1}$$

be any cubic form in n variables with integral coefficients. Then if $n \geqslant 32$ *there exist integers* $x_1, \ldots, x_n$, *not all* 0, *such that*

$$C(x_1, \ldots, x_n) = 0. \tag{2}$$

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It is known that the assertion of the theorem would be false for $n < 10$ (see Mordell 1937).

The method of proof is in principle that of Hardy and Littlewood. An investigation based on this method was made by Tartakowsky (1935), but his work appears to be incomplete, and in any case relates only to 'general' forms. Apart from this work, almost all the existing applications of the Hardy–Littlewood method are to additive problems, and depend from the outset on the fact that the exponential sum appropriate to the problem is a product of sums, each in one variable. In the present work we have to deal with exponential sums containing a general cubic form in n variables. The method used here for estimating such sums is a development of that which I have introduced elsewhere (Davenport 1956, 1958) in connexion with exponential sums containing a general quadratic form.

The Hardy–Littlewood method, when it is effective, yields an asymptotic formula for the number of solutions in integers of a given equation, possibly with subsidiary conditions. The main term in the formula is the product of two factors. One of these can be regarded as measuring the density of the solutions in the real field; the other is itself an infinite product, with one factor for every prime p, and this factor can be regarded as measuring the density of the solutions in the p-adic field. Such asymptotic formulae arise in the present work, but here they have no absolute validity, for in establishing them we make use of the hypothesis that the equation in question has no non-zero solution. One point that emerges more clearly in the present application of the Hardy–Littlewood method than in previous ones is that the equation in question must have not merely a non-zero solution but a non-singular solution, both in the real field and in every p-adic field, in order that the asymptotic formula shall be significant.

The solubility of (2) in every p-adic field, provided $n \geqslant 10$, has been established recently, in different ways, by Demyanov (1950) and Lewis (1952); the former, however, assumes that $p \neq 3$. In §2 I give my own proof, constructed before I was aware of these papers. Though somewhat long, it has certain features of interest. None of the three proofs seems to extend easily to forms of higher degree, though it is known that there is always a p-adic solution if n is sufficiently large (Brauer 1945).

The proof of the theorem is continued in §§3 to 7, each of which (like §2) is largely self-contained, and is completed in §8. §3 is concerned with general cubic exponential sums, and forms the most difficult and delicate part of the investigation.

The synthesis in §8 is accomplished in several stages. In the first stage, it is proved that $C(x_1, \ldots, x_n)$, where $n \geqslant 32$, either represents zero or represents a cubic form of the type

$$C_1(x_1, \ldots, x_{n-8}) + dy^3.$$

The process of splitting off cubes is continued in the subsequent stages, which make increasing use of methods appropriate to Waring's Problem. The final stage is that of proving the solubility of

$$d_1 y_1^3 + \ldots + d_8 y_8^3 = 0,$$

and this can be carried out by an adaptation of known methods (Davenport 1939).

It may be of interest to remark that the analogous problem for forms of even degree $2r$ $(r > 1)$ presents a difficulty of formulation, quite apart from the formidable difficulties of proof. Mr H. P. F. Swinnerton-Dyer has shown me an example of a form of degree 6 in any number of variables which does not represent zero, even though the corresponding equation

has a non-singular solution in the real field and in every p-adic field. This example is provided by the form
$$3(x_1^2+\ldots+x_r^2)^3+4(x_{r+1}^2+\ldots+x_s^2)^3-5(x_{s+1}^2+\ldots+x_n^2)^3,$$
where $r<s<n$. The insolubility of the equation $3x^3+4y^3-5z^3=0$ in integers (not all 0) was proved by Selmer (1951).

2. The p-adic problem

(*a*) Let
$$C(\mathbf{x})=C(x_1,\ldots,x_n)=\sum_i\sum_j\sum_k c_{ijk}x_ix_jx_k \tag{1}$$
be a cubic form with integral coefficients c_{ijk} which are symmetrical in the three suffixes. We shall suppose that $C(\mathbf{x})$ is non-degenerate, that is, that $C(\mathbf{x})$ is not equivalent to a cubic form in fewer than n variables. This is obviously true if $C(\mathbf{x})$ does not represent zero.

Our aim in this section is to establish the existence, for every prime p, of a positive integer l with the following property: for every sufficiently large ν, the congruence
$$C(x_1,\ldots,x_n)\equiv 0 \pmod{p^\nu}$$
has a solution for which $\quad \partial C/\partial x_i \not\equiv 0 \pmod{p^l}$ for some i.

This is the meaning, expressed in elementary terms, of the assertion that the equation $C(\mathbf{x})=0$ has a non-singular solution in the p-adic field (see §1). We shall also prove that there is an upper bound for l which depends only on a certain arithmetical invariant of C, and is independent of p. This, however, is not essential for the application later.

The first two lemmas apply in principle to forms of any degree, and may be classical.

Let $N=\frac{1}{2}n(n+1)$. Let $\mathscr{C}$ denote the matrix of n rows and N columns whose general element is c_{ijk}, where i indicates the row and the pair j, k, with $j\leqslant k$, indicates the column, all these N pairs being arranged in some fixed order. Let Δ denote a typical determinant of order n formed from any n columns of $\mathscr{C}$, the number of possible determinants being $\binom{N}{n}$.

Definition. *Let $h(C)$ denote the highest common factor of all the determinants Δ, if they are not all 0, and in the latter case let $h(C)=0$.*

Lemma 2·1. *Let*
$$x_i'=\sum_{r=1}^{n} q_{ir}x_r \quad (1\leqslant i\leqslant n)$$
be a linear transformation with integral coefficients q_{ir} of determinant $q\neq 0$, and let
$$C(x_1,\ldots,x_n)=C'(x_1',\ldots,x_n')$$
identically. Then $h(C)$ is divisible by $qh(C')$.

Proof. The coefficients in the two forms C and C' are related by
$$c_{rst}=\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} q_{ir}q_{js}q_{kt}c_{ijk}'.$$
In defining the matrix $\mathscr{C}$ above, we chose a one-to-one correspondence between pairs j, k with $1\leqslant j\leqslant k\leqslant n$ and integers μ with $1\leqslant\mu\leqslant N$. Thus the general element of $\mathscr{C}'$ is $c_{ijk}'=c_{i\mu}'$, say, where $i=1,\ldots,n$ and $\mu=1,\ldots,N$. Similarly, representing the pair s, t with $s\leqslant t$ by ν, the general element of $\mathscr{C}$ is $c_{rst}=c_{r\nu}$. Put
$$u_{\mu\nu}=\begin{cases} q_{js}q_{kt} & \text{if } \ j=k,\\ q_{js}q_{kt}+q_{ks}q_{jt} & \text{if } \ j<k,\end{cases}$$

where μ denotes the pair j, k and ν the pair s, t. Then the relation between the two sets of coefficients can be written

$$c_{r\nu} = \sum_{i=1}^{n} q_{ir} \sum_{\mu=1}^{N} c'_{i\mu} u_{\mu\nu},$$

where $r = 1, \ldots, n$ and $\nu = 1, \ldots, N$. In matrix notation, this is

$$\mathscr{C} = \mathscr{Q}^{\mathrm{T}} \mathscr{C}' \mathscr{U},$$

where $\mathscr{Q} = q_{ir}$ is an $n \times n$ matrix and $\mathscr{U} = u_{\mu\nu}$ is an $N \times N$ matrix, and T denotes the transpose.

Let Δ be the determinant formed from the columns $\nu_1, \ldots, \nu_n$ of $\mathscr{C}$, or symbolically:

$$\Delta = (\det \mathscr{C})^{1, \ldots, n}_{\nu_1, \ldots, \nu_n}.$$

Since the matrix $\mathscr{Q}$ has determinant q, it follows that

$$\pm \Delta = q(\det \mathscr{C}' \mathscr{U})^{1, \ldots, n}_{\nu_1, \ldots, \nu_n}.$$

By a well-known result (MacDuffee 1946, theorem 7·9) we have

$$(\det \mathscr{C}' \mathscr{U})^{1, \ldots, n}_{\nu_1, \ldots, \nu_n} = \sum_{\rho_1, \ldots, \rho_n} (\det \mathscr{C}')^{1, \ldots, n}_{\rho_1, \ldots, \rho_n} (\det \mathscr{U})^{\rho_1, \ldots, \rho_n}_{\nu_1, \ldots, \nu_n},$$

where the summation is over all $\binom{N}{n}$ selections of $\rho_1, \ldots, \rho_n$ from $1, \ldots, N$ without regard to order.

In each term of the sum, the first factor is one of the determinants Δ' of order n that can be formed from $\mathscr{C}'$, and the second factor is an integer. Hence the sum is divisible by $h(C')$, and it follows that Δ is divisible by $qh(C')$. This proves the result.

Corollary. *$h(C)$ is an arithmetical invariant of C, that is, it has the same value for any two equivalent forms.*

Proof. If C and C' are equivalent forms, the lemma applies with $q = 1$, and shows that $h(C)$ is divisible by $h(C')$. Similarly, $h(C')$ is divisible by $h(C)$, whence the result.

Lemma 2·2. *If $C(\mathbf{x})$ is non-degenerate then $h(C) \neq 0$.*

Proof. If $h(C) = 0$ then all determinants of order n formed from $\mathscr{C}$ vanish, that is, the n rows of $\mathscr{C}$ are linearly dependent. Thus there exist $p_1, \ldots, p_n$, not all 0, such that

$$\sum_{i=1}^{n} p_i c_{ijk} = 0$$

for all j, k; and we can take $p_1, \ldots, p_n$ to be integers with highest common factor 1. Since

$$\frac{1}{3} \frac{\partial C}{\partial x_i} = \sum_j \sum_k c_{ijk} x_j x_k,$$

we have

$$p_1 \frac{\partial C}{\partial x_1} + \ldots + p_n \frac{\partial C}{\partial x_n} = 0$$

identically in $x_1, \ldots, x_n$. It is well known that there exists an $n \times n$ matrix of integers p_{ir}, of determinant ± 1, such that $p_{in} = p_i$ for $i = 1, \ldots, n$. Putting

$$x_i = \sum_{r=1}^{n} p_{ir} y_r,$$

we have $\partial C/\partial y_n = 0$ identically, and consequently $C(\mathbf{x})$ is equivalent to a form in $y_1, \ldots, y_{n-1}$ and is degenerate.

The converse of the last lemma is also true, for the above argument is reversible, but it will not be needed.

(*b*) In the subsequent work, up to lemma 2·8, it will be convenient to deal with cubic forms in which the coefficients are integers but are not necessarily divisible by 3 (for terms such as $x_1^2x_2$) or 6 (for terms such as $x_1x_2x_3$), as is implied by the representation (1) and the condition of symmetry. We therefore consider now cubic forms of the type

$$C(x_1, \ldots, x_n) = \sum_{i\leqslant j\leqslant k} d_{ijk}x_ix_jx_k, \tag{2}$$

where the d_{ijk} are integers.

Definition. *Let p be a prime and l a positive integer. We say that $C(x)$ has the property $\mathscr{A}(p^l)$ if there exist integers $x_1, \ldots, x_n$ such that*

$$C(x_1, \ldots, x_n) \equiv 0 \pmod{p^{2l-1}}, \tag{3}$$

$$\partial C/\partial x_1 \equiv \ldots \equiv \partial C/\partial x_n \equiv 0 \pmod{p^{l-1}}, \tag{4}$$

$$\partial C/\partial x_i \not\equiv 0 \pmod{p^l} \textit{ for some } i. \tag{5}$$

For brevity we express (4) and (5) by

$$p^{l-1} \,\|\, (\partial C/\partial x_1, \ldots, \partial C/\partial x_n). \tag{6}$$

It is plain that the property $\mathscr{A}(p^l)$ is arithmetically invariant: if it holds for one form then it holds for any equivalent form. Indeed this remains true with a wider definition of equivalence, admitting linear transformations with any determinant that is relatively prime to p.

Lemma 2·3. *Suppose $C(\mathbf{x})$ has the property $\mathscr{A}(p^l)$. Then for any $\nu \geqslant 0$ the congruence*

$$C(x_1, \ldots, x_n) \equiv 0 \pmod{p^{2l-1+\nu}} \tag{7}$$

has at least $p^{(n-1)\nu}$ solutions, mutually incongruent to the modulus $p^{l+\nu}$, and each satisfying (6).

Proof. We proceed by induction on ν; when $\nu = 0$ the assertion is merely that of $\mathscr{A}(p^l)$. We assume the result for a particular value of ν and deduce the corresponding result with $\nu+1$ in place of ν.

For any variables $x_1, \ldots, x_n, u_1, \ldots, u_n$ we have the identical congruence

$$C(\mathbf{x}+p^{l+\nu}\mathbf{u}) \equiv C(\mathbf{x}) + p^{l+\nu}(u_1\partial C/\partial x_1 + \ldots + u_n\partial C/\partial x_n) \quad \pmod{p^{2l+2\nu}}.$$

This is obvious if $p > 3$, but it holds also if $p = 2$ or 3. For, by (2),

$$\partial^2C/\partial x_1^2 = 6d_{111}x_1 + 2d_{112}x_2 + \ldots,$$

$$\partial^3C/\partial x_1^3 = 6d_{111}, \quad \partial^3C/\partial x_1^2\partial x_2 = 2d_{112},$$

and it is easily seen that the apparent denominators arising from the factors $1/2!$ and $1/3!$ in the two further terms of the Taylor expansion all disappear.

We take $x_1, \ldots, x_n$ to be any one of the $p^{(n-1)\nu}$ solutions of (7) and (6), the various choices being mutually incongruent modulo $p^{l+\nu}$. We can put

$$C(\mathbf{x}) = ap^{2l-1+\nu}, \quad \partial C/\partial x_i = D_ip^{l-1},$$

where $a, D_1, \ldots, D_n$ are integers and $D_i \not\equiv 0 \pmod{p}$ for some i. The congruence

$$C(\mathbf{x}+p^{l+\nu}\mathbf{u}) \equiv 0 \pmod{p^{2l+\nu}}$$

holds if $$a+D_1u_1+\ldots+D_nu_n\equiv 0 \pmod{p}.$$

This has p^{n-1} solutions in $\mathbf{u}$, mutually incongruent modulo p.

Thus, corresponding to each $\mathbf{x}$, we obtain p^{n-1} values of $\mathbf{y}=\mathbf{x}+p^{l+\nu}\mathbf{u}$. These satisfy

$$C(\mathbf{y})\equiv 0 \pmod{p^{2l+\nu}},\quad \mathbf{y}\equiv\mathbf{x} \pmod{p^{l+\nu}}.$$

From the latter it follows that each $\mathbf{y}$ satisfies (6). We obtain altogether $p^{(n-1)(\nu+1)}$ values for $\mathbf{y}$, and they are mutually incongruent modulo $p^{l+\nu+1}$. This is the desired result.

(*c*) Lemma 2·4. *If* $n\geqslant 4$ *and* $C(\mathbf{x})$ *does not have the property* $\mathscr{A}(p)$, *then* $C(\mathbf{x})$ *is equivalent to a form of the type*

$$C'(x_1,x_2,x_3)+pC''(x_1,\ldots,x_n). \tag{8}$$

Proof. By a theorem of Chevalley† there is a solution of $C(\mathbf{x})\equiv 0 \pmod{p}$ other than $\mathbf{x}\equiv 0$, since the number of variables exceeds the degree of the congruence. As $C(\mathbf{x})$ does not have the property $\mathscr{A}(p)$, we must have $\partial C/\partial x_i\equiv 0 \pmod{p}$ for all i. After a suitable integral unimodular transformation, we can take the solution in question to be

$$x_1=1, x_2=\ldots=x_n=0.$$

Then $C(\mathbf{x})$ has the form

$$C(\mathbf{x})=apx_1^3+px_1^2(b_2x_2+\ldots+b_nx_n)+x_1(b_{22}x_2^2+\ldots+b_{23}x_2x_3+\ldots)+C_{n-1}(x_2,\ldots,x_n),$$

the coefficients of $x_1^2x_2,\ldots,x_1^2x_n$ being $\equiv 0 \pmod{p}$ because they are the values of $\partial C/\partial x_2,\ldots,\partial C/\partial x_n$ at the solution.

If $b_{22},\ldots,b_{23},\ldots$, are not all $\equiv 0 \pmod{p}$, we can choose $x_2,\ldots,x_n$ so that

$$b_{22}x_2^2+\ldots+b_{23}x_2x_3+\ldots\not\equiv 0 \pmod{p},$$

by taking values of the type $1,0,\ldots,0$ or of the type $1,1,0,\ldots,0$. We can then choose x_1 so that

$$x_1(b_{22}x_2^2+\ldots+b_{23}x_2x_3+\ldots)+C_{n-1}(x_2,\ldots,x_n)\equiv 0 \pmod{p},$$

and this gives a solution of $C\equiv 0 \pmod{p}$ with $\partial C/\partial x_1\not\equiv 0 \pmod{p}$, contrary to hypothesis.

We can therefore suppose that $b_{22},\ldots,b_{23},\ldots$ are all $\equiv 0 \pmod{p}$. Thus $C(\mathbf{x})$ is equivalent to

$$px_1Q_n(x_1,\ldots,x_n)+C_{n-1}(x_2,\ldots,x_n),$$

where Q_n is a quadratic form (though not necessarily with even coefficients for the product terms).

If $n\geqslant 5$ we can put $x_1=0$ and apply the same argument to $C_{n-1}(x_2,\ldots,x_n)$, since this form also cannot have the property $\mathscr{A}(p)$. Thus C_{n-1} is equivalent to

$$px_2Q_{n-1}(x_2,\ldots,x_n)+C_{n-2}(x_3,\ldots,x_n).$$

The process continues until we reach $C_3(x_{n-2},x_{n-1},x_n)$, to which Chevalley's theorem does not apply. Hence $C(\mathbf{x})$ is equivalent to a form of the type

$$p(x_1Q_n+x_2Q_{n-1}+\ldots+x_{n-3}Q_4)+C_3(x_{n-2},x_{n-1},x_n).$$

Reversing the order of writing the variables, we obtain a form of the type (8).

It may be noted that the resulting form $C''(x_1,\ldots,x_n)$ in (8) contains no terms in x_1,x_2,x_3 only. But this fact will not be needed in the sequel.

† See, for example, Davenport (1952, pp. 55–57).

Lemma 2·5. *If, in the result of lemma* 2·4, *the form*

$$C''(0,0,0,x_4,\ldots,x_n)$$

in $x_4,\ldots,x_n$ *has the property* $\mathscr{A}(p^\lambda)$, *then* $C(\mathbf{x})$ *has the property* $\mathscr{A}(p^l)$ *for some* $l \leqslant \lambda+1$.

Proof. By lemma 2·3 with $\nu = 1$, there exist $x_4,\ldots,x_n$ such that

$$C''(0,0,0,x_4,\ldots,x_n) \equiv 0 \pmod{p^{2\lambda}},$$

$$p^{\lambda-1} \,||\, (\partial C''/\partial x_4,\ldots,\partial C''/\partial x_n).$$

Hence, for the values 0, 0, 0, $x_4,\ldots,x_n$ we have $C \equiv 0 \pmod{p^{2\lambda+1}}$ and

$$p^\lambda \,||\, (\partial C/\partial x_4,\ldots,\partial C/\partial x_n).$$

If, for these values of $x_1,\ldots,x_n$, we define l by

$$p^{l-1} \,||\, (\partial C/\partial x_1,\ldots,\partial C/\partial x_n),$$

then $l \leqslant \lambda+1$ and $C(\mathbf{x})$ has the property $\mathscr{A}(p^l)$.

Lemma 2·6. *If* $n \geqslant 10$ *and* $C(\mathbf{x})$ *does not have any of the properties* $\mathscr{A}(p), \mathscr{A}(p^2), \mathscr{A}(p^3)$, *then it is equivalent to a form of the type*

$$C^*(x_1,\ldots,x_9,px_{10},\ldots,px_n). \tag{9}$$

Proof. In the expression (8) for a form equivalent to C, which we denote again by C, we put $x_i = py_i$ for $i = 1,2,3$. This gives

$$C(py_1,py_2,py_3,x_4,\ldots,x_n) = p^3C'(y_1,y_2,y_3) + {}_{,}pC''(py_1,py_2,py_3,x_4,\ldots,x_n).$$

Ignoring multiples of p^3, we have

$$C(py_1,py_2,py_3,x_4,\ldots,x_n) \equiv p^2C_{1,2}(y_1,y_2,y_3 \,|\, x_4,\ldots,x_n) + pC''(0,0,0,x_4,\ldots,x_n) \pmod{p^3}, \tag{10}$$

where $C_{1,2}$ denotes a form which is of the first degree in y_1,y_2,y_3 and the second degree in $x_4,\ldots,x_n$.

By lemma 2·5, the form $C''(0,0,0,x_4,\ldots,x_n)$ in $x_4,\ldots,x_n$ does not have either of the properties $\mathscr{A}(p), \mathscr{A}(p^2)$. We apply lemma 2·4 to this form, and put $x_i = py_i$ for $i = 4,5,6$ in the result. Neglecting multiples of p^2, we obtain

$$C''(0,0,0,py_4,py_5,py_6,x_7,\ldots,x_n) \equiv pC^{(3)}(0,\ldots,0,x_7,\ldots,x_n) \pmod{p^2}. \tag{11}$$

Further, by lemma 2·5, the form $C^{(3)}(0,\ldots,0,x_7,\ldots,x_n)$ in $x_7,\ldots,x_n$ does not have the property $\mathscr{A}(p)$.

Putting $x_i = py_i$ for $i = 4,5,6$ in (10), and using (11), we obtain a result which can be written

$$C(py_1,\ldots,py_6,x_7,\ldots,x_n) \equiv p^2(y_1Q_1+y_2Q_2+y_3Q_3) + p^2C^{(3)}(0,\ldots,0,x_7,\ldots,x_n) \pmod{p^3}, \tag{12}$$

where Q_1, Q_2, Q_3 are quadratic forms in $x_7,\ldots,x_n$. It will be noted that y_4,y_5,y_6 do not appear on the right.

Suppose that one of the quadratic forms, say Q_1, is not identically $\equiv 0 \pmod{p}$. Then there exist $x_7,\ldots,x_n$ for which $Q_1 \not\equiv 0 \pmod{p}$, and we can choose y_1, y_2, y_3 so that

$$y_1Q_1+y_2Q_2+y_3Q_3+C^{(3)}(0,\ldots,0,x_7,\ldots,x_n) \equiv 0 \pmod{p}.$$

This gives
$$C(py_1,\ldots,py_6,x_7,\ldots,x_n) \equiv 0 \pmod{p^3},$$

the values of y_4, y_5, y_6 being arbitrary. Also

$$(\partial/\partial y_1)\,C(py_1, \ldots, py_6, x_7, \ldots, x_n) \equiv p^2 Q_1 \not\equiv 0 \pmod{p^3}.$$

Taking $x_i = py_i$ for $i \leqslant 6$, and noting that $\partial/\partial x_1 = p^{-1}\partial/\partial y_1$, we have values of $x_1, \ldots, x_n$ for which $C \equiv 0 \pmod{p^3}$ and $\partial C/\partial x_1 \not\equiv 0 \pmod{p^2}$. This contradicts the hypothesis that $C(\mathbf{x})$ does not have either of the properties $\mathscr{A}(p), \mathscr{A}(p^2)$.

Thus Q_1, Q_2, Q_3 are all identically $\equiv 0 \pmod{p}$, and (12) becomes

$$C(py_1, \ldots, py_6, x_7, \ldots, x_n) \equiv p^2 C^{(3)}(0, \ldots, 0, x_7, \ldots, x_n) \pmod{p^3}.$$

Finally, we apply lemma 2·4 to the form $C^{(3)}$ in $x_7, \ldots, x_n$, which (as already noted) does not have the property $\mathscr{A}(p)$. We obtain

$$C^{(3)}(0, \ldots, 0, x_7, \ldots, x_n) \equiv C^{(4)}(x_7, x_8, x_9) \pmod{p}.$$

Putting $x_i = py_i$ for $i = 7, 8, 9$, we get

$$C(py_1, \ldots, py_9, x_{10}, \ldots, x_n) \equiv 0 \pmod{p^3},$$

and this holds identically in $y_1, \ldots, y_9, x_{10}, \ldots, x_n$. Denoting the form on the left by

$$p^3 C^*(y_1, \ldots, y_9, x_{10}, \ldots, x_n),$$

we have the identity $\quad C(x_1, \ldots, x_n) = C^*(x_1, \ldots, x_9, px_{10}, \ldots, px_n).$

Lemma 2·7. *Suppose that $n \geqslant 10$. If, in the result of lemma 2·6, the form $C^*(x_1, \ldots, x_n)$ has the property $\mathscr{A}(p^\lambda)$, then the form $C(x_1, \ldots, x_n)$ has the property $\mathscr{A}(p^l)$ for some $l \leqslant \lambda + 3$.*

Proof. By lemma 2·3 with $\nu = 3$, there exist $y_1, \ldots, y_9, x_{10}, \ldots, x_n$ such that

$$C^*(y_1, \ldots, y_9, x_{10}, \ldots, x_n) \equiv 0 \pmod{p^{2\lambda+2}}, \quad p^{\lambda-1} \,||\, (\partial C^*/\partial y_1, \ldots.\, \partial C^*/\partial x_n).$$

Since $\quad C(py_1, \ldots, py_9, x_{10}, \ldots, x_n) = p^3 C^*(y_1, \ldots, y_9, x_{10}, \ldots, x_n)$

identically, we have $\quad C(py_1, \ldots, py_9, x_{10}, \ldots, x_n) \equiv 0 \pmod{p^{2\lambda+5}},$

and one at least of $\partial C/\partial y_1, \ldots, \partial C/\partial x_n$ is not divisible by $p^{\lambda+3}$. Taking $x_i = py_i$ for $i \leqslant 9$, we see that one at least of $\partial C/\partial x_1, \ldots, \partial C/\partial x_n$ is not divisible by $p^{\lambda+3}$, whence the result.

(*d*) We are now in a position to prove the main result of this section, namely:

Lemma 2·8. *Any non-degenerate cubic form in at least* 10 *variables has the property $\mathscr{A}(p^l)$ for every prime p and a suitable l depending on p. There is an upper bound for l depending on the cubic form but not on p.*

Proof. Suppose $C(x_1, \ldots, x_n)$ is a cubic form, of the type (2), which does not have any of the properties $\mathscr{A}(p), \mathscr{A}(p^2), \ldots, \mathscr{A}(p^{3m})$, where m is a positive integer. By lemma 2·6, this form is equivalent to a form of the type (9). This implies that there is a linear transformation

$$x_i' = \sum_{r=1}^{n} q_{ir} x_r \quad (1 \leqslant i \leqslant n),$$

with integral coefficients and determinant p^{n-9}, which transforms $C(x_1, \ldots, x_n)$ into another form, say $C^{(1)}(x_1', \ldots, x_n')$ with integral coefficients. By lemma 2·7, the form $C^{(1)}$ does not have any of the properties $\mathscr{A}(p), \ldots, \mathscr{A}(p^{3m-3})$. By repetition, it follows that there is a linear transformation with integral coefficients and determinant $p^{(n-9)m}$ which transforms $C(\mathbf{x})$ into a form $C^{(m)}(\mathbf{y})$ with integral coefficients.

We now revert to the considerations of lemmas 2·1 and 2·2, and in order to do so we consider the forms $6C(\mathbf{x})$ and $6C^{(m)}(\mathbf{y})$, which can be written in the notation (1). It follows from lemma 2·1 that $h(6C)$ is divisible by $p^{(n-9)m}$. Further, $h(6C)$ is a positive integer by lemma 2·2. Thus

$$(n-9)\,m \leqslant (\log h(6C))/\log p \leqslant (\log h(6C))/\log 2,$$

and this gives an upper bound for m which is independent of p. The proof is complete.

3. General cubic exponential sums

(*a*) Let

$$\Gamma(\mathbf{x}) = \Gamma(x_1, \ldots, x_n) = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \gamma_{ijk} x_i x_j x_k \tag{1}$$

be a cubic form with arbitrary real coefficients γ_{ijk} which are symmetrical in the three suffixes. Let $\mathscr{B}$ be any box in n dimensional space of the form

$$x'_j \leqslant x_j < x''_j \quad (j = 1, \ldots, n). \tag{2}$$

We shall suppose that

$$0 < x''_j - x'_j \leqslant 1, \tag{3}$$

but this is merely for convenience. Let P be a large positive integer, and let

$$S = \sum_{\mathbf{x} \text{ in } P\mathscr{B}} \mathrm{e}\,(\Gamma(x_1, \ldots, x_n)), \tag{4}$$

where the summation is over the integer points in the box $P\mathscr{B}$ defined by $Px'_j \leqslant x_j < Px''_j$, and where $\mathrm{e}\,(\alpha) = \mathrm{e}^{2\pi i\alpha}$. The only restriction imposed on n in the present section is that $n > 1$.

The obvious estimate for S is† $|S| \ll P^n$, and no more is true if, for example, the γ_{ijk} are all integers. Our aim in this section is to investigate the consequences of the hypothesis that

$$|S| \gg P^{n-\kappa}, \tag{5}$$

where κ is a fixed positive number, small compared with n. It is natural to expect that the coefficients γ_{ijk} will then satisfy many approximate linear relations with integral coefficients.

Let

$$B_j(\mathbf{x} \mid \mathbf{y}) = \sum_{i=1}^{n} \sum_{k=1}^{n} \gamma_{ijk} x_i y_k \quad (j = 1, \ldots, n). \tag{6}$$

It will be convenient to write $\mathbf{B} = (B_1, \ldots, B_n)$ and

$$|\mathbf{B}| = \max\,(|B_1|, \ldots, |B_n|)$$

for any point (or vector) $\mathbf{B}$. We denote by $\|\alpha\|$ the difference between any real number α and the nearest integer, taken positively, and we write

$$\|\boldsymbol{\alpha}\| = \max\,(\|\alpha_1\|, \ldots, \|\alpha_n\|)$$

for any point $\boldsymbol{\alpha}$.

We shall prove that the hypothesis (5) implies the existence, for suitable exponents θ, of a positive integer m, depending only on n, κ, θ, with the following property: there exist a non-zero integer point $\mathbf{x}$ and m linearly independent integer points $\mathbf{y}^{(1)}, \ldots, \mathbf{y}^{(m)}$ such that

$$|\mathbf{x}| < P^{\theta}, \quad |\mathbf{y}^{(r)}| < P^{\theta}, \tag{7}$$

$$\|6\mathbf{B}(\mathbf{x} \mid \mathbf{y}^{(r)})\| < P^{-3+2\theta+\delta}, \tag{8}$$

† The notation $\ll$ or $\gg$ indicates an inequality with an unspecified constant factor. Any number which does not depend on P or on the γ_{ijk} is deemed to be a constant.

for $r = 1, \ldots, m$. Here δ is a positive number which can be taken arbitrarily small, but is independent of P. In the later application, θ will be given values between about $\frac{1}{2}$ and $\frac{3}{4}$.

The inequalities (8) can be regarded as a system of mn approximate linear relations connecting the γ_{ijk}, and together with (7) these are of the form which is most convenient for application later. The larger m can be taken, the more effective is the result. We shall ultimately prove, in lemma 3·7, that m can be taken to be about $n - 2\kappa/\theta$, provided that $1 > \theta > 4\kappa/n$.

Although we are concerned primarily with the sum S defined in (4), it is important to observe that the results apply equally if $\Gamma(\mathbf{x})$ is a non-homogeneous cubic polynomial. As we shall see, any quadratic or linear terms in $\Gamma(\mathbf{x})$ disappear at the first step, in lemma 3·1. We make use of this remark in lemma 4·3.

(*b*) We base the argument on an inequality for $|S|^4$ which is obtained by following the first step of Weyl's method for estimating exponential sums in one variable.

LEMMA 3·1. *We have*

$$|S|^4 \ll P^n \sum_{|\mathbf{x}|<P} \sum_{|\mathbf{y}|<P} \prod_{j=1}^{n} \min(P, \| 6B_j(\mathbf{x} \mid \mathbf{y}) \|^{-1}).$$

Proof. We have

$$|S|^2 = \sum_{\mathbf{z} \text{ in } P\mathscr{B}} \sum_{\mathbf{z}' \text{ in } P\mathscr{B}} \mathrm{e}(\Gamma(\mathbf{z}') - \Gamma(\mathbf{z})) = \sum_{\mathbf{z} \text{ in } P\mathscr{B}} \sum_{\mathbf{y} \text{ in } P\mathscr{B}-\mathbf{z}} \mathrm{e}(\Gamma(\mathbf{z}+\mathbf{y}) - \Gamma(\mathbf{z})).$$

The box $P\mathscr{B} - \mathbf{z}$ is contained in the cube $|\mathbf{y}| < P$, by the supposition (3). Hence

$$|S|^2 \leqslant \sum_{|\mathbf{y}|<P} \Big| \sum_{\mathbf{z} \text{ in } \mathscr{R}(\mathbf{y})} \mathrm{e}(\Gamma(\mathbf{z}+\mathbf{y}) - \Gamma(\mathbf{z})) \Big|,$$

where $\mathscr{R}(\mathbf{y})$ denotes the common part of $P\mathscr{B}$ and $P\mathscr{B} - \mathbf{y}$. This is itself a box, with edges less than P in length.

Applying the same argument to the inner sum over $\mathbf{z}$, we see that its square does not exceed

$$\sum_{|\mathbf{x}|<P} \Big| \sum_{\mathbf{z} \text{ in } \mathscr{S}(\mathbf{x},\mathbf{y})} \mathrm{e}(\Gamma(\mathbf{z}+\mathbf{x}+\mathbf{y}) - \Gamma(\mathbf{z}+\mathbf{x}) - \Gamma(\mathbf{z}+\mathbf{y}) + \Gamma(\mathbf{z})) \Big|,$$

where $\mathscr{S}(\mathbf{x}, \mathbf{y})$ denotes the common part of the boxes $\mathscr{R}(\mathbf{y})$ and $\mathscr{R}(\mathbf{y}) - \mathbf{x}$. Plainly this is again a box with edges less than P in length.

We have

$$\Gamma(\mathbf{z}+\mathbf{x}+\mathbf{y}) - \Gamma(\mathbf{z}+\mathbf{x}) - \Gamma(\mathbf{z}+\mathbf{y}) + \Gamma(\mathbf{z}) = 6 \sum_{i,j,k} \gamma_{ijk} x_i y_k z_j + \phi = 6 \sum_j B_j(\mathbf{x} \mid \mathbf{y}) z_j + \phi,$$

where ϕ is independent of $\mathbf{z}$ and all summations over i, j, k are from 1 to n. The well-known inequality

$$\Big| \sum_z \mathrm{e}(\lambda z) \Big| \ll \min(P, \| \lambda \|^{-1}),$$

where the summation is over any set of $\ll P$ consecutive integers, gives

$$\Big| \sum_{\mathbf{z} \text{ in } \mathscr{S}(\mathbf{x},\mathbf{y})} \mathrm{e}\big(6 \sum_j B_j(\mathbf{x} \mid \mathbf{y}) z_j\big) \Big| \ll \prod_{j=1}^{n} \min(P, \| 6B_j(\mathbf{x} \mid \mathbf{y}) \|^{-1}).$$

Substitution in the previous result, combined with Cauchy's inequality, gives the conclusion stated.

We note that, as remarked in (*a*), any quadratic or linear terms in $\Gamma(\mathbf{x})$ would disappear, since they would give rise only to terms independent of $\mathbf{z}$ in the second-difference expression considered above.

LEMMA 3·2. *Suppose* (5) *holds. Let N denote the number of distinct pairs of integer points* $\mathbf{x}$, $\mathbf{y}$ *satisfying*

$$|\mathbf{x}|<P, |\mathbf{y}|<P, \| 6\mathbf{B}(\mathbf{x}\,|\,\mathbf{y}) \| < P^{-1}. \tag{9}$$

Then

$$N \gg P^{2n-4\kappa}(\log P)^{-n}. \tag{10}$$

Proof. Let $N(\mathbf{x})$ denote, for any $\mathbf{x}$, the number of distinct integer points $\mathbf{y}$ satisfying (9), so that

$$N = \sum_{|\mathbf{x}|<P} N(\mathbf{x}).$$

Let $f(\alpha)$ denote temporarily the fractional part of a real number α. Then, for any integer point $\mathbf{x}$ and any integers $r_1, \ldots, r_n$ satisfying $0 \leqslant r_j < P$, the inequalities

$$P^{-1}r_j \leqslant f(6B_j(\mathbf{x}\,|\,\mathbf{y})) < P^{-1}(r_j+1) \quad (j = 1, \ldots, n)$$

cannot hold for more than $N(\mathbf{x})$ integer points $\mathbf{y}$ with each of $y_1, \ldots, y_n$ lying in some prescribed interval of length P. For if $\mathbf{y}'$ is one solution of the inequalities, and $\mathbf{y}$ denotes the general solution, then

$$\| 6B_j(\mathbf{x}\,|\,\mathbf{y}-\mathbf{y}') \| < P^{-1} \quad (j = 1, \ldots, n),$$

and $|\mathbf{y}-\mathbf{y}'| < P$. Thus the number of possibilities for $\mathbf{y}$ is at most $N(\mathbf{x})$.

Dividing the summation over $|\mathbf{y}| < P$ into 2^n parts, for each of which $y_1, \ldots, y_n$ run through intervals of length P, we obtain

$$\sum_{|\mathbf{y}|<P} \prod_{j=1}^{n} \min(P, \| 6B_j(\mathbf{x}\,|\,\mathbf{y}) \|^{-1}) \ll N(\mathbf{x}) \sum_{r_1=0}^{P-1} \ldots \sum_{r_n=0}^{P-1} \prod_{j=1}^{n} \min\left(P, \frac{P}{r_j}, \frac{P}{P-r_j-1}\right) \ll N(\mathbf{x})\,(P\log P)^n.$$

Applying this in the result of lemma 3·1, we obtain

$$|S|^4 \ll P^n (P\log P)^n \sum_{|\mathbf{x}|<P} N(\mathbf{x}),$$

and the conclusion (10) follows from (5).

(*c*) We now recall some propositions concerning linear inequalities which follow from known results in the geometry of numbers. Let $L_1(\mathbf{u}), \ldots, L_n(\mathbf{u})$ be n real linear forms in n variables $u_1, \ldots, u_n$, say

$$L_j(\mathbf{u}) = \sum_{k=1}^{n} \lambda_{jk} u_k \quad (j = 1, \ldots, n). \tag{11}$$

They need not be independent, and may even vanish identically. The inequalities

$$|u_j| < A, \quad |L_j(\mathbf{u}) - t_j| < A^{-1} \quad (j = 1, \ldots, n) \tag{12}$$

define, for any $A > 0$ a convex body (a parallelepiped) in the $2n$ dimensional space in which the co-ordinates are $u_1, \ldots, u_n, t_1, \ldots, t_n$. Let $M_1, \ldots, M_{2n}$ denote the $2n$ successive minima of this body, in the sense of Minkowski, relative to the $2n$-dimensional lattice consisting of all points with integral co-ordinates. We have

$$0 < M_1 \leqslant M_2 \leqslant \ldots \leqslant M_{2n}. \tag{13}$$

The effectiveness of the theory of successive minima, in the present connexion, depends on the linear forms $L_1, \ldots, L_n$ being *symmetrical*; by this we mean that the coefficients λ_{jk} in (11) satisfy

$$\lambda_{jk} = \lambda_{kj} \quad (1 \leqslant j \leqslant k \leqslant n). \tag{14}$$

LEMMA 3·3. *Suppose that $A>1$. For $0<Z\leqslant 1$, let $U(Z)$ denote the number of distinct n dimensional integer points* $\mathbf{u}$ *satisfying*

$$|\mathbf{u}|<ZA, \quad ||L(\mathbf{u})||<ZA^{-1}. \tag{15}$$

Then, if the linear forms $L_1, \ldots, L_n$ are symmetrical, and if $0<Z_1<Z_2\leqslant 1$, we have

$$U(Z_1)\geqslant (Z_1/Z_2)^n U(Z_2). \tag{16}$$

Proof. We recall first some results proved in Davenport (1958). The first is that

$$M_n \ll 1 \ll M_{n+1}. \tag{17}$$

This is a deduction from a theorem of Mahler, and is conditional on the linear forms $L_1, \ldots, L_n$ being symmetrical. The second is an upper bound for $U(Z)$. If $Z<M_1$, the only solution of (15) is $\mathbf{u}=0$, for by definition M_1 is the least number such that the inequalities

$$|\mathbf{u}|\leqslant M_1 A, \quad |\mathbf{L}(\mathbf{u})-\mathbf{t}|\leqslant M_1 A^{-1}$$

have a solution in integers $u_1, \ldots, u_n, t_1, \ldots, t_n$, not all 0. Thus $U(Z)=1$ for $Z<M_1$. For $Z\geqslant M_1$, there exists $g\,(1\leqslant g\leqslant n)$ such that

$$M_g \leqslant Z \ll M_{g+1}, \tag{18}$$

and it was proved in Davenport (1958) that

$$U(Z)\ll \frac{Z^g}{M_1 M_2 \ldots M_g}. \tag{19}$$

The complementary inequality

$$U(Z)\gg \frac{Z^g}{M_1 M_2 \ldots M_g} \tag{20}$$

is also true for $Z\geqslant M_1$, and its proof is in fact rather simpler than that of (19). Let $\mathbf{u}^{(r)}$, $\mathbf{t}^{(r)}$ be the rth minimal point for the body (12) in $2n$-dimensional space, so that

$$|\mathbf{u}^{(r)}|\leqslant M_r A, \, |\mathbf{L}(\mathbf{u}^{(r)})-\mathbf{t}^{(r)}|\leqslant M_r A^{-1} \tag{21}$$

for $r=1, \ldots, 2n$. These points are linearly independent in the $2n$-dimensional space. Consider the points $\mathbf{u}$ in n dimensions given by

$$\mathbf{u}=m_1\mathbf{u}^{(1)}+\ldots+m_g\mathbf{u}^{(g)}, \tag{22}$$

where $m_1, \ldots, m_g$ take all integral values satisfying

$$|m_r|<\tfrac{1}{2}g^{-1}ZM_r^{-1} \quad (r=1, \ldots, g). \tag{23}$$

By (21), every such point satisfies

$$|\mathbf{u}|<\tfrac{1}{2}ZA, \quad |\mathbf{L}(\mathbf{u})-(m_1\mathbf{t}^{(1)}+\ldots+m_g\mathbf{t}^{(g)}|<\tfrac{1}{2}ZA^{-1},$$

and so satisfies (15). Thus $U(Z)$ is at least equal to the number of distinct n-dimensional points $\mathbf{u}$ so obtained. If two of the points $\mathbf{u}$ coincided, then their difference, which is given by (22) and (23) without the factor $\frac{1}{2}$ in (23), would be 0. This would imply

$$|m_1\mathbf{t}^{(1)}+\ldots+m_g\mathbf{t}^{(g)}|<ZA^{-1},$$

and so

$$m_1\mathbf{t}^{(1)}+\ldots+m_g\mathbf{t}^{(g)}=0.$$

But this, together with $\mathbf{u} = 0$, would contradict the linear independence of the points $\mathbf{u}^{(r)}$, $\mathbf{t}^{(r)}$ in $2n$-dimensional space. Thus the points $\mathbf{u}$ constructed above are distinct, and since the numbers on the right of (23) are all $\gg 1$, we obtain (20).

The result (16) is now a simple deduction. Supposing first $Z_1 > M_1$, we determine g as in (18) for Z_1, and determine h similarly for Z_2. Then $h \geqslant g$. By (19) applied to Z_2 and (20) applied to Z_1, we have

$$\frac{U(Z_2)}{U(Z_1)} \ll \frac{Z_2^h}{Z_1^g M_{g+1} \dots M_h} \ll \left(\frac{Z_2}{Z_1}\right)^h \ll \left(\frac{Z_2}{Z_1}\right)^n.$$

The possibilities $Z_1 \leqslant M_1 \leqslant Z_2$ and $Z_2 \leqslant M_1$ present no difficulty; indeed (19) and (20) remain valid for $Z \leqslant M_1$ if one makes the convention that then $g = 0$.

(*d*) Lemma 3·4. *Suppose* (5) *holds. Let* θ *satisfy* $0 < \theta < 1$. *Let* N_2 *denote the number of distinct pairs of integer points* $\mathbf{x}$, $\mathbf{y}$ (*each in n dimensions*) *satisfying*

$$|\mathbf{x}| < P^{\theta}, \quad |\mathbf{y}| < P^{\theta}, \quad \|6\mathbf{B}(\mathbf{x}\,|\,\mathbf{y})\| < P^{-3+2\theta}. \tag{24}$$

Then
$$N_2 \gg P^{2n\theta - 4\kappa}(\log P)^{-n}. \tag{25}$$

Proof. We make two separate applications of lemma 3·3; in the first we regard the bilinear forms $B_j(\mathbf{x}\,|\,\mathbf{y})$ as linear forms in the variables $\mathbf{y}$ for each particular $\mathbf{x}$, and in the second we regard them as linear forms in the variables $\mathbf{x}$ for each particular $\mathbf{y}$.

Let $N(\mathbf{x})$ denote, as in the proof of lemma 3·2, the number of integer points $\mathbf{y}$ satisfying (9), so that

$$\sum_{|\mathbf{x}| < P} N(\mathbf{x}) \gg P^{2n - 4\kappa}(\log P)^{-n}. \tag{26}$$

For each $\mathbf{x}$ we apply lemma 3·3 with $\mathbf{u} = \mathbf{y}$ and with

$$L_j(\mathbf{u}) = 6B_j(\mathbf{x}\,|\,\mathbf{y}).$$

The coefficients λ_{jk} in (11) are given by

$$\lambda_{jk} = 6\sum_{i=1}^{n} \gamma_{ijk} x_i,$$

and the condition (14) is satisfied. We take

$$A = P, \quad Z_1 = P^{-1+\theta}, \quad Z_2 = 1.$$

The inequalities (15) with $Z = 1$ are the same as the second and third inequalities in (9), so that $U(Z_2) = N(\mathbf{x})$. The inequalities (15) with $Z = P^{-1+\theta}$ become

$$|\mathbf{y}| < P^{\theta}, \quad \|6\mathbf{B}(\mathbf{x}\,|\,\mathbf{y})\| < P^{-2+\theta};$$

denoting the number of solutions of these in $\mathbf{y}$ by $N_1(\mathbf{x})$, we have $U(Z_1) = N_1(\mathbf{x})$. By (16),

$$N_1(\mathbf{x}) \gg P^{-n(1-\theta)} N(\mathbf{x}). \tag{27}$$

Let N_1 denote the number of distinct pairs $\mathbf{x}$, $\mathbf{y}$ of integer points satisfying

$$|\mathbf{x}| < P, \quad |\mathbf{y}| < P^{\theta}, \quad \|6\mathbf{B}(\mathbf{x}\,|\,\mathbf{y})\| < P^{-2+\theta}. \tag{28}$$

Then
$$N_1 = \sum_{|\mathbf{x}| < P} N_1(\mathbf{x}) \gg P^{n + n\theta - 4\kappa}(\log P)^{-n}$$
by (26), (27).

Let $N_1'(\mathbf{y})$ denote, for any $\mathbf{y}$, the number of integer points $\mathbf{x}$ satisfying (28), so that

$$\sum_{|\mathbf{y}| < P^{\theta}} N_1'(\mathbf{y}) = N_1 \gg P^{n + n\theta - 4\kappa}(\log P)^{-n}. \tag{29}$$

For each $\mathbf{y}$ we apply lemma 3·3 with $\mathbf{u} = \mathbf{x}$ and with $L_j(\mathbf{u}) = 6B_j(\mathbf{x} \mid \mathbf{y})$. The condition (14) is again satisfied. We take

$$A = P^{\frac{1}{2}(3-\theta)}, \quad Z_1 = P^{-\frac{3}{2}(1-\theta)}, \quad Z_2 = P^{-\frac{1}{2}(1-\theta)}.$$

The inequalities (15) with $Z = Z_2$ are the same as the first and third of (28), so that $U(Z_2) = N_1'(\mathbf{y})$. The inequalities (15) with $Z = Z_1$ become the first and third inequalities of (24), and denoting the number of solutions of these in $\mathbf{x}$ by $N_2(\mathbf{y})$ we have $U(Z_1) = N_2(\mathbf{y})$. By (16),

$$N_2(\mathbf{y}) \gg P^{-n(1-\theta)} N_1'(\mathbf{y}).$$

Finally, we have

$$N_2 = \sum_{|\mathbf{y}|<P^\theta} N_2(\mathbf{y}) \gg P^{-n(1-\theta)} \sum_{|\mathbf{y}|<P^\theta} N_1'(\mathbf{y}) \gg P^{2n\theta-4\kappa}(\log P)^{-n},$$

by (29). This proves the desired result.

It should be noted that the result (25) is of no value unless $2n\theta - 4\kappa > n\theta$, for the trivial solutions of (24) with $\mathbf{x} = 0$ or $\mathbf{y} = 0$ are in number $\gg P^{n\theta}$.

(*e*) In lemma 3·4 we have established the existence, under the hypothesis (5), of a substantial number of pairs $\mathbf{x}$, $\mathbf{y}$ of integer points satisfying (24). We now wish to select from them as many pairs as possible with a common $\mathbf{x}$ and with various points $\mathbf{y}$ that are linearly independent. It is easy to prove that we can find as many as $n - 4\kappa/\theta$ such pairs, but we shall improve this to $n - 2\kappa/\theta$.

The problem just stated can be formulated in general terms as follows. Suppose there is a law of association between certain integer points $\mathbf{x}$ and certain integer points $\mathbf{y}$, of a symmetrical nature. Suppose we know a lower bound for the number of pairs of associated integer points satisfying $|\mathbf{x}| < P^\theta$, $|\mathbf{y}| < P^\theta$. For how large a value of m can we assert that, among these points, there is one point $\mathbf{x}$ which is associated to m linearly independent points $\mathbf{y}$? In the present case, the association is prescribed by the inequality

$$\|\, 6\mathbf{B}(\mathbf{x} \mid \mathbf{y}) \,\| < P^{-3+2\theta}, \tag{30}$$

and in the subsequent work we make use of the fact that this association is almost linear in $\mathbf{x}$ and $\mathbf{y}$ separately. In particular, if $\mathbf{x}$, $\mathbf{y}'$ are associated and $\mathbf{x}$, $\mathbf{y}''$ are associated, then $\mathbf{x}$, $\mathbf{y}' - \mathbf{y}''$ are almost associated, in the sense that they satisfy (30) if a factor 2 is inserted on the right. Such a factor, provided it has to be inserted only a bounded number of times, is harmless, and in effect our law of association can be regarded as linear.

We base the investigation on the following elementary lemma, which is essentially an instance of Dirichlet's compartment principle.

LEMMA 3·5. *Let $\mathscr{Y}$ be any set of at least T distinct integer points $\mathbf{y}$ in the n dimensional cube $|\mathbf{y}| < Y$, where $Y > 1$ and $T > 1$. Suppose there are at most m linearly independent points in the set $\mathscr{Y}$. Then*

$$T \ll Y^m. \tag{31}$$

Further, if W is a positive integer satisfying $1 < W \leqslant T$, there exist W distinct points in $\mathscr{Y}$, say $\mathbf{y}^{(1)}, \ldots, \mathbf{y}^{(W)}$, such that

$$|\, \mathbf{y}^{(1)} - \mathbf{y}^{(w)} \,| \ll Y W^{1/m} T^{-1/m} \tag{32}$$

for $w = 2, \ldots, W$.

Proof. Since both the conclusions are weakened by increasing m, we can take m to be precisely the maximum number of linearly independent points in $\mathscr{Y}$. We shall suppose that $m < n$; if $m = n$ the proof is similar but simpler.

By the definition of m, there exist $n-m$ independent homogeneous linear equations satisfied by the co-ordinates $y_1, \ldots, y_n$ of every point in $\mathscr{Y}$. Let one of them be

$$\alpha_1^{(1)} y_1 + \ldots + \alpha_n^{(1)} y_n = 0;$$

we can suppose without loss of generality that

$$|\alpha_1^{(1)}| \geqslant |\alpha_j^{(1)}| \quad (2 \leqslant j \leqslant n).$$

A second independent linear equation can always be taken in the form

$$\alpha_2^{(2)} y_2 + \ldots + \alpha_n^{(2)} y_n = 0,$$

and we can suppose without loss of generality that

$$|\alpha_2^{(2)}| \geqslant |\alpha_j^{(2)}| \quad (3 \leqslant j \leqslant n).$$

Continuing in this way, we obtain $n-m$ such equations, and they express $y_1, \ldots, y_{n-m}$ as homogeneous linear forms in $y_{n-m+1}, \ldots, y_n$ with bounded coefficients.

Since there are at most $2Y+1$ possibilities for each of $y_{n-m+1}, \ldots, y_n$ and these determine $y_1, \ldots, y_{n-m}$, the estimate (31) is obvious.

To prove the second result, we take L to be the greatest positive integer satisfying $WL^m \leqslant T$. Divide each of the intervals

$$-Y < y_j < Y \quad (n-m < j \leqslant n)$$

into L equal parts. The set $\mathscr{Y}$ falls into L^m parts, and one of these must contain at least W distinct points. Denoting these points by $\mathbf{y}^{(w)}$, we have

$$|y_j^{(1)} - y_j{}^{(w)}| < 2Y/L \quad (n-m < j \leqslant n)$$

for $w = 2, \ldots, W$. The points $\mathbf{y}^{(1)} - \mathbf{y}^{(w)}$, though not necessarily in $\mathscr{Y}$, satisfy the $n-m$ homogeneous linear equations described earlier, and consequently

$$|y_j^{(1)} - y_j^{(w)}| \ll Y/L \quad (1 \leqslant j \leqslant n)$$

for $w = 2, \ldots, W$. Since $L \gg (T/W)^{1/m}$, we obtain (32).

(*f*) Definition. *Let* n, κ, θ $(0 < \theta < 1)$, $\delta (>0)$ *be given independently of* P, *and suppose that* $n\theta > 4\kappa$. *Define m to be the greatest positive integer with the property that there exist a non-zero integer point* $\mathbf{x}$ *and m linearly independent integer points* $\mathbf{y}^{(1)}, \ldots, \mathbf{y}^{(m)}$ *such that*

$$|\mathbf{x}| < P^\theta, \quad |\mathbf{y}^{(r)}| < P^\theta, \quad \|6\mathbf{B}(\mathbf{x}, \mathbf{y}^{(r)})\| < P^{-3+2\theta+\delta} \tag{33}$$

for $r = 1, \ldots, m$. Note that these inequalities are the same as (24) except for the last exponent, which has been increased by δ (to allow for the fact that the association between $\mathbf{x}$ and $\mathbf{y}$ defined by such an inequality is not precisely linear). Note also that the existence of $m (\geqslant 1)$ is ensured by the condition $n\theta > 4\kappa$, for there exist $\mathbf{x} \neq 0$ and $\mathbf{y} \neq 0$ satisfying (24), and *a fortiori* (33).

In what follows, ϵ denotes an arbitrarily small positive number, independent of P.

Lemma 3·6. *Suppose that*

$$0 < \xi \leqslant \theta, \quad 0 < \eta \leqslant \theta, \quad 0 < B \leqslant P^{-3+2\theta+\delta}. \tag{34}$$

Suppose there exist more than $P^{\lambda+\epsilon}$ distinct pairs $\mathbf{x}$, $\mathbf{y}$ *of integer points, neither of them* 0, *which satisfy*

$$|\mathbf{x}|<P^{\xi}, \quad |\mathbf{y}|<P^{\eta}, \quad \|6\mathbf{B}(\mathbf{x}\,|\,\mathbf{y})\|<B. \tag{35}$$

Suppose that $$\lambda>n\xi, \quad \lambda>m\eta. \tag{36}$$

Then $$\lambda<n\xi+m\eta. \tag{37}$$

Further, there exist more than $P^{n\xi}$ distinct pairs of integer points, neither of them 0, *which satisfy*

$$|\mathbf{x}|<P^{\xi}, \quad |\mathbf{y}|<P^{\eta-(\lambda-n\xi)/m}, \quad \|6\mathbf{B}(\mathbf{x}\,|\,\mathbf{y})\|<2B. \tag{38}$$

Proof. Let $T(\mathbf{x})$ denote, for each $\mathbf{x}\neq 0$, the number of integer points $\mathbf{y}\neq 0$ satisfying (35). We apply lemma 3·5 to this set of points; in doing so we can take m to be the number defined above, since the inequalities (35), with (34), imply (33). It follows from (31) that

$$T(\mathbf{x})\ll P^{m\eta}. \tag{39}$$

By hypothesis, $$\sum_{|\mathbf{x}|<P^{\xi}} T(\mathbf{x})>P^{\lambda+\epsilon}, \tag{40}$$

whence $P^{\lambda+\epsilon}\ll P^{n\xi+m\eta}$, giving (37).

Let c be a positive constant such that $T(\mathbf{x})<cP^{m\eta}$. For $r=0,1,2,\ldots$ let X_r denote the number of integer points $\mathbf{x}\neq 0$ in $|\mathbf{x}|<P^{\xi}$ for which

$$cP^{m\eta}2^{-r-1}\leqslant T(\mathbf{x})<cP^{m\eta}\,2^{-r}. \tag{41}$$

It suffices to take $\ll\log P$ values of r to include all $\mathbf{x}$ for which $T(\mathbf{x})\geqslant 1$. By (40),

$$\sum_{r} \mathrm{c}P^{m\eta}2^{-r}\,X_r>P^{\lambda+\epsilon}.$$

Hence there exists a value of r for which

$$P^{m\eta}\,2^{-r}X_r\gg P^{\lambda+\epsilon}(\log P)^{-1}. \tag{42}$$

Define ρ by $$P^{\rho}=P^{m\eta-\frac{1}{2}\epsilon}\,2^{-r},$$

so that $\rho\leqslant m\eta-\frac{1}{2}\epsilon$. By (42), $$X_r\gg P^{\lambda-\rho+\frac{1}{3}\epsilon}, \tag{43}$$

and since $X_r\ll P^{n\xi}$ trivially, it follows that $\rho>\lambda-n\xi+\frac{1}{4}\epsilon$. Note that the exponent in (43) is positive, since $\lambda>m\eta>\rho$.

We now have X_r integer points $\mathbf{x}\neq 0$, to each of which there correspond $T(\mathbf{x})$ integer points $\mathbf{y}\neq 0$, such that the inequalities (35) are satisfied by each pair $\mathbf{x}$, $\mathbf{y}$. We apply to the set of points $\mathbf{y}$ (corresponding to a particular $\mathbf{x}$) the second part of lemma 3·5, with

$$Y=P^{\eta}, \quad T=T(\mathbf{x})\gg P^{\rho+\frac{1}{2}\epsilon}, \quad W=[P^{\rho-\lambda+n\xi}]+1. \tag{44}$$

The condition $1<W<T(\mathbf{x})$ is satisfied, since $0<\lambda-n\xi<\rho$. It follows that there exist W distinct points $\mathbf{y}$ in the set, say $\mathbf{y}^{(1)},\ldots,\mathbf{y}^{(W)}$, with the property that

$$|\mathbf{y}^{(1)}-\mathbf{y}^{(w)}|\ll P^{\eta}\,P^{(\rho-\lambda+n\xi)/m}P^{-(\rho+\frac{1}{2}\epsilon)/m}=P^{\eta-(\lambda-n\xi)/m-\frac{1}{2}\epsilon/m},$$

for $w=2,\ldots,W$. Hence to each of the X_r points $\mathbf{x}$ there correspond $W-1$ distinct integer points $\mathbf{y}\neq 0$ satisfying $$|\mathbf{y}|<P^{\eta-(\lambda-n\xi)/m}, \quad \|6\mathbf{B}(\mathbf{x}\,|\,\mathbf{y})\|<2B.$$

All the resulting pairs $\mathbf{x}$, $\mathbf{y}$ satisfy (38), and their number is

$$(W-1)\,X_r\gg P^{\rho-\lambda+n\xi}\,P^{\lambda-\rho+\frac{1}{3}\epsilon}=P^{n\xi+\frac{1}{3}\epsilon},$$

by (43), (44). This completes the proof.

Lemma 3·7. *Let n, κ, θ $(0<\theta<1), \delta(>0)$ be fixed independently of P, and suppose that*

$$n\theta > 4\kappa. \tag{45}$$

Then, under the hypothesis (5), *we have*

$$m \geqslant n - 2\kappa/\theta. \tag{46}$$

Proof. The argument is based on repeated applications of lemma 3·6. We can suppose that $m < n$, since otherwise (46) holds.

We first apply lemma 3·6 with

$$\xi = \theta, \quad \eta = \theta, \quad B = P^{-3+2\theta}, \tag{47}$$

so that the inequalities (35) become the same as (24). By lemma 3·4, these inequalities have more than $P^{2n\theta-4\kappa-\epsilon}$ solutions, and this remains true if we count only solutions with both $\mathbf{x} \neq 0$ and $\mathbf{y} \neq 0$, since the number omitted is $\ll P^{n\theta}$ and $n\theta < 2n\theta - 4\kappa - \epsilon$ by (45) if ϵ is sufficiently small. Thus, in lemma 3·6 we can take

$$\lambda = 2n\theta - 4\kappa - 2\epsilon. \tag{48}$$

The first of the hypotheses (36) is satisfied, by (45), and the second is satisfied *a fortiori* since $\xi = \eta$ and $m < n$. Hence (37) holds, and we also obtain a result about the number of solutions of (38).

This result provides the basis for a second application of lemma 3·6. It is convenient to interchange $\mathbf{x}$ and $\mathbf{y}$ in (38) before proceeding. Thus, putting

$$\xi_1 = \eta - (\lambda - n\xi)/m, \quad \eta_1 = \xi, \quad B_1 = 2B, \quad \lambda_1 = n\xi - \epsilon, \tag{49}$$

the result in question is that the inequalities

$$|\mathbf{x}| < P^{\xi_1}, \quad |\mathbf{y}| < P^{\eta_1}, \quad \|6\mathbf{B}(\mathbf{x}\,|\,\mathbf{y})\| < B_1 \tag{50}$$

have more than $P^{\lambda_1+\epsilon}$ solutions with $\mathbf{x} \neq 0$, $\mathbf{y} \neq 0$.

The first of the conditions corresponding to (36), namely $\lambda_1 > n\xi_1$, is equivalent (since $\xi = \eta$) to $\lambda - n\xi > m\epsilon/n$, and is satisfied by (45) if ϵ is sufficiently small. The second condition, namely $\lambda_1 > m\eta_1$, is satisfied since $n > m$. Hence lemma 3·6 is applicable a second time, giving

$$\lambda_1 < n\xi_1 + m\eta_1$$

and giving further a similar new starting point with numbers $\xi_2, \eta_2, B_2, \lambda_2$.

The process (as long as it can be continued) is inductive: at the general stage we obtain

$$\lambda_q < n\xi_q + m\eta_q, \tag{51}$$

and we get new numbers $\xi_{q+1}, \eta_{q+1}, B_{q+1}, \lambda_{q+1}$ given by

$$\xi_{q+1} = \eta_q - (\lambda_q - n\xi_q)/m, \quad \eta_{q+1} = \xi_q, \quad B_{q+1} = 2B_q, \tag{52}$$

$$\lambda_{q+1} = n\xi_q - \epsilon. \tag{53}$$

The process continues as long as the conditions corresponding to (36) are satisfied. We shall limit ourselves, however, to a number of stages which is bounded independently of P; this ensures that

$$B_q = 2^q B = 2^q P^{-3+2\theta} < P^{-3+2\theta+\delta},$$

and consequently the inequalities satisfied by $\mathbf{x}$ and $\mathbf{y}$ at each stage imply the inequalities (33) used in the definition of m.

We shall now prove that the process can be continued for any prescribed number of stages (the number being independent of P) provided ϵ is sufficiently small. We note first that, by (52) and (53),

$$\xi_{q+1} = \xi_{q-1} - (n\xi_{q-1} - \epsilon - n\xi_q)/m,$$

that is

$$\xi_q - \xi_{q+1} = \omega(\xi_{q-1} - \xi_q) - \epsilon/m, \tag{54}$$

where

$$\omega = -1 + n/m > 0. \tag{55}$$

Since $\xi - \xi_1 > 0$ it follows by induction that $\xi_q - \xi_{q+1} > 0$ for any prescribed number of stages, provided ϵ is sufficiently small.

The conditions corresponding to (36) which must be satisfied for the step from q to $q+1$ are

$$\lambda_q > n\xi_q, \quad \lambda_q > m\eta_q.$$

The former is equivalent to

$$n\xi_{q-1} - \epsilon > n(\eta_{q-1} - (\lambda_{q-1} - n\xi_{q-1})/m),$$

and so is equivalent to

$$m(\xi_{q-2} - \xi_{q-1}) < n(\xi_{q-2} - \xi_{q-1}) - \epsilon(m+n)/n.$$

Since $\xi_{q-2} - \xi_{q-1} > 0$ and $m \leqslant n-1$, this is satisfied if ϵ is sufficiently small. The second condition is equivalent to

$$n\xi_{q-1} - \epsilon > m\xi_{q-1},$$

and is satisfied if ϵ is sufficiently small.

By induction from (54), we have

$$\xi_q - \xi_{q+1} = \omega^q(\xi - \xi_1) - K(q)\ \epsilon,$$

where $K(q)$ is independent of ϵ. By summation,

$$\xi - \xi_{q+1} = (1 + \omega + \omega^2 + \ldots + \omega^q)\ (\xi - \xi_1) - K_1(q)\,\epsilon,$$

with a similar meaning for $K_1(q)$. Since $\xi_{q+1} > 0$ by (51) and (52), we have

$$\xi > (1 + \omega + \ldots + \omega^q)\ (\xi - \xi_1) - K_1(q)\,\epsilon.$$

Since $\xi - \xi_1 = \eta - \xi_1 = (\lambda - n\xi)/m > 0$ by (49) and (36), this relation cannot hold for large q if $\omega \geqslant 1$. If $\omega < 1$, so that $\omega \leqslant 1 - 1/m$ by (55), then on taking q sufficiently large and ϵ sufficiently small we obtain

$$\xi \geqslant (1-\omega)^{-1}\,(\xi - \xi_1) = (1-\omega)^{-1}\,(\lambda - n\xi)/m,$$

whence

$$(2m - n)\,\xi \geqslant \lambda - n\xi = 2n\theta - 4\kappa - 2\epsilon - n\xi$$

by (48), and since $\xi = \theta$ this gives (46).

4. Two particular types of exponential sum

(*a*) We now apply the results of § 3 to two types of exponential sum. Both types are defined in terms of a fixed cubic form

$$C(\mathbf{x}) = C(x_1, \ldots, x_n) = \sum_i \sum_j \sum_k c_{ijk} x_i x_j x_k \tag{1}$$

with integral coefficients, which we suppose not to represent zero. The first type is simply the sum S of § 3 with $\Gamma(\mathbf{x}) = \alpha C(\mathbf{x})$, where α is a real number. Thus the sum in question is

$$S(\alpha) = \sum_{\mathbf{x} \text{ in } P\mathscr{B}} \mathrm{e}\,(\alpha C(\mathbf{x})), \tag{2}$$

where $\mathscr{B}$ is a fixed n-dimensional box and P is a large positive integer. We shall obtain estimates for $|S(\alpha)|$ depending upon suitable rational approximations to α.

The second type of exponential sum is purely arithmetical in character. Let a, q be integers with $(a, q) = 1$ and $q > 0$, and let $\mathbf{l} = (l_1, \ldots, l_n)$ be a set of n integers. Then the sum in question is

$$S_{a,q}(\mathbf{l}) = \sum_{\mathbf{x}} e\left(\frac{a}{q}C(\mathbf{x}) + \frac{l_1x_1 + \ldots + l_nx_n}{q}\right), \tag{3}$$

where the summation for $x_1, \ldots, x_n$ is over n complete sets of residues (mod q). A particular case of the sum is

$$S_{a,q} = S_{a,q}(0). \tag{4}$$

The estimates obtained for both $S(\alpha)$ and $S_{a,q}(\mathbf{l})$ will be of a conditional character; they will be proved on the supposition that $C(\mathbf{x})$ does not split in the manner defined below.

(b) We recall that a cubic form $C(x_1, \ldots, x_n)$ is said to *represent* a cubic form $C'(u_1, \ldots, u_s)$, where $s \leqslant n$, if there exist n linear forms in s variables, say

$$x_i = \sum_{t=1}^{s} p_{it} u_t \quad (1 \leqslant i \leqslant n),$$

with integral coefficients and with rank s, such that

$$C(x_1, \ldots, x_n) = C'(u_1, \ldots, u_s)$$

identically. It is plain that if C does not represent zero then neither does C', for

$$x_1 = \ldots = x_n = 0 \quad \text{implies} \quad u_1 = \ldots = u_s = 0.$$

Definition. *A cubic form $C(x_1, \ldots, x_n)$ will be said to split with remainder m if it represents a form in $m+1$ variables of the type*

$$a_0u_0^3 + C_1(u_1, \ldots, u_m). \tag{5}$$

Lemma 4·1. *Let $C(x_1, \ldots, x_n)$ be the cubic form (1), and let*[†]

$$B_j(\mathbf{x} \mid \mathbf{y}) = \sum_i \sum_k c_{ijk} x_i y_k \quad \textit{for} \quad j = 1, \ldots, n.$$

Suppose there exist a non-zero integer point $\mathbf{z}$ and m linearly independent integer points $\mathbf{y}^{(1)}, \ldots, \mathbf{y}^{(m)}$, where $1 \leqslant m \leqslant n$, such that

$$B_j(\mathbf{z} \mid \mathbf{y}^{(r)}) = 0 \quad \textit{for} \quad 1 \leqslant j \leqslant n, \quad 1 \leqslant r \leqslant m. \tag{6}$$

Then $m \leqslant n-1$ and $C(\mathbf{x})$ splits with remainder m.

Proof. Suppose first that $\mathbf{z}$ is linearly dependent on $\mathbf{y}^{(1)}, \ldots, \mathbf{y}^{(m)}$, so that

$$\mathbf{z} = v_1\mathbf{y}^{(1)} + \ldots + v_m\mathbf{y}^{(m)}$$

for some rational $v_1, \ldots, v_m$. This, of course, must be so if $m = n$. Then (6) implies that $B_j(\mathbf{z} \mid \mathbf{z}) = 0$, but then

$$C(\mathbf{z}) = \sum_j z_j B_j(\mathbf{z} \mid \mathbf{z}) = 0,$$

contrary to hypothesis.

We can now suppose that $\mathbf{z}, \mathbf{y}^{(1)}, \ldots, \mathbf{y}^{(m)}$ are linearly independent. Thus the linear forms

$$x_i = u_0 z_i + u_1 y_i^{(1)} + \ldots + u_m y_i^{(m)} \quad (1 \leqslant i \leqslant n)$$

[†] There is a minor inconsistency of notation compared with §3; the two meanings given to $B_j(\mathbf{x} \mid \mathbf{y})$ differ by a factor α.

have rank $m+1$. In the resulting cubic form in $u_0, u_1, \ldots, u_m$, the coefficient of $u_0^2 u_r$ is

$$3\sum_i\sum_j\sum_k c_{ijk} z_i z_j y_k^{(r)} = 3\sum_j z_j B_j(\mathbf{z} \mid \mathbf{y}^{(r)}) = 0$$

for $1 \leqslant r \leqslant m$; and the coefficient of $u_0 u_r u_s$ is 3 or 6 times

$$\sum_i\sum_j\sum_k c_{ijk} z_i y_j^{(r)} y_k^{(s)} = \sum_j y_j^{(r)} B_j(\mathbf{z} \mid \mathbf{y}^{(s)}) = 0$$

for $1 \leqslant r \leqslant m$, $1 \leqslant s \leqslant n$. Hence the cubic form in $u_0, u_1, \ldots, u_m$ is of the type (5), as asserted.

(*c*) Let δ be a small positive number, independent of P, as in § 3 (*f*).

Lemma 4·2. *Let κ and θ be fixed numbers satisfying*

$$0 < \theta < 1, \tag{7}$$

$$0 < 4\kappa < n\theta. \tag{8}$$

Let $C(\mathbf{x})$ be a fixed cubic form with integral coefficients which does not split with remainder m, where m is the least integer satisfying

$$m \geqslant n - 2\kappa/\theta. \tag{9}$$

Then, for any real α, either

$$|S(\alpha)| < P^{n-\kappa}, \tag{10}$$

or there exists a rational approximation a'/q' to α satisfying

$$1 \leqslant q' < P^{2\theta+\delta}, \tag{11}$$

$$|\alpha - a'/q'| < q'^{-1} P^{-3+2\theta+\delta}. \tag{12}$$

Proof. Suppose (10) false, so that the hypothesis (5) of § 3 holds. The hypotheses of lemma 3·7 are satisfied. By that lemma and the definition of m in § 3 (*f*), there exists an integer m satisfying (9) with the following property: there exist a non-zero integer point $\mathbf{x}$ and m linearly independent integer points $\mathbf{y}^{(1)}, \ldots, \mathbf{y}^{(m)}$ such that

$$|\mathbf{x}| < P^{\theta}, \quad |\mathbf{y}^{(r)}| < P^{\theta}, \quad \| 6\alpha \mathbf{B}(\mathbf{x} \mid \mathbf{y}^{(r)}) \| < P^{-3+2\theta+\delta}, \tag{13}$$

for $r = 1, \ldots, m$. The inequalities (13) are the same as (33) of § 3 but with $\mathbf{B}$ replaced by $\alpha\mathbf{B}$, corresponding to the fact that $\Gamma(\mathbf{x}) = \alpha C(\mathbf{x})$.

It is impossible that

$$B_j(\mathbf{x} \mid \mathbf{y}^{(r)}) = 0 \quad \text{for} \quad j = 1, \ldots, n; \quad r = 1, \ldots, m,$$

for then, by lemma 4·1, $C(\mathbf{x})$ would split with remainder m, contrary to hypothesis. Thus for some j and some r, $B_j(\mathbf{x} \mid \mathbf{y}^{(r)})$ is a non-zero integer. The last of the inequalities (13) gives

$$|6\alpha B_j(\mathbf{x} \mid \mathbf{y}^{(r)}) - t| < P^{-3+2\theta+\delta}$$

for some integer t. Take a'/q' to be the rational number $t/6B_j(\mathbf{x} \mid \mathbf{y}^{(r)})$ reduced to its lowest terms. Then

$$|q'\alpha - a'| < P^{-3+2\theta+\delta}$$

and

$$1 \leqslant q' \leqslant 6\,|B_j(\mathbf{x} \mid \mathbf{y}^{(r)})| \ll P^{2\theta}$$

by (13). Thus a'/q' satisfies (11) and (12).

It should be noted that lemma 4·2 cannot yield any useful information unless θ is taken to be less than $\frac{3}{4}$; for if $\theta \geqslant \frac{3}{4}$, every α has a rational approximation satisfying (11) and (12).

LEMMA 4·3. *Let $C(\mathbf{x})$ be a fixed cubic form in n variables with integral coefficients which does not represent zero. Let κ be a fixed number satisfying*

$$0<\kappa<\tfrac{1}{8}n. \tag{14}$$

Suppose $C(\mathbf{x})$ does not split with remainder m, where m is the least integer satisfying

$$m\geqslant n-4\kappa. \tag{15}$$

Then
$$|S_{a,q}(\mathbf{l})|<q^{n-\kappa} \tag{16}$$
for all, a, q, $\mathbf{l}$ subject to $(a, q)=1$.

Proof. As remarked in §3(a), the presence of the linear terms $(l_1x_1+\ldots+l_nx_n)/q$ in the definition of $S_{a,q}(\mathbf{l})$ does not affect the results of §3. Thus $S_{a,q}(\mathbf{l})$ is essentially the same as the sum $S(\alpha)$ with the specializations $\alpha=a/q$ and $P=q$, and with the box $\mathscr{B}$ taken to be $0\leqslant x_j<1$, so that the summation over $P\mathscr{B}$ is over $0\leqslant x_j<q$. Plainly a/q cannot have any rational approximation satisfying (11) and (12) if we take $\theta=\frac{1}{2}(1-\delta)$, and then the last alternative in lemma 4·2 cannot occur. The condition (8) of lemma 4·2 is satisfied if δ is sufficiently small, by (14). The condition that $C(\mathbf{x})$ does not split, with remainder m satisfying (9), also holds if δ is sufficiently small and $\theta=\frac{1}{2}(1-\delta)$.

If q is sufficiently large, lemma 4·2 is applicable and gives the desired result (16). If, on the other hand, q is bounded in terms of n, κ and the coefficients of $C(\mathbf{x})$, the estimate (16) asserts no more than the trivial estimate $|S_{a,q}(\mathbf{l})|\leqslant q^n$. This proves the lemma.

(e) The general nature of lemma 4·3 is at first sight a little strange. The sum $S_{a,q}(\mathbf{l})$ depends only on the coefficients of $C(\mathbf{x})$ and the numbers $\mathbf{l}$ to the modulus q, whereas the splitting of $C(\mathbf{x})$ is an absolute property. It would appear more natural to have the estimation of $S_{a,q}(\mathbf{l})$ related to some property of $C(\mathbf{x})$ relative to the modulus q. But it will be seen that we have taken $C(\mathbf{x})$ to be a fixed form, independent of q, and this explains the apparent anomaly. (What we actually supposed, when estimating $|B_j(\mathbf{x}\,|\,\mathbf{y}^{(r)})|$ in the proof of lemma 4·2, was that the coefficients of $C(\mathbf{x})$ are absolutely bounded.) It would be of interest to have an effective estimate for $S_{a,q}(\mathbf{l})$ which was related only to the properties of $C(\mathbf{x})$ modulo q.

5. Approximation on major arcs

(a) Let $C(\mathbf{x})$ be again a fixed cubic form in n variables, with integral coefficients, which does not represent zero. We now study the sum $S(\alpha)$, defined in (2) of §4, for values of α satisfying

$$|\alpha-a/q|<q^{-1}P^{-2-\delta}, \tag{1}$$

where
$$(a,q)=1,\quad 1\leqslant q\leqslant P^{1-\delta}. \tag{2}$$

These values of α will include the 'major arcs' in the subsequent applications of the Hardy–Littlewood method in §8. We write $\beta=\alpha-a/q$.

In the present section, it is convenient to denote the box $P\mathscr{B}$ which occurs in the definition of $S(\alpha)$ by

$$A_j<x_j<B_j\quad(j=1,\ldots,n), \tag{3}$$

and to suppose that $A_j-\frac{1}{2}$, $B_j-\frac{1}{2}$ are integers. The aim, as usual in this connexion, is to approximate to $S(\alpha)$ by

$$q^{-n}S_{a,q}I(\beta), \tag{4}$$

where $S_{a,q}$ is defined by (4) of § 4 and where

$$I(\beta) = \int_{A_1}^{B_1} \dots \int_{A_n}^{B_n} \mathrm{e}\,(\beta C(\xi_1, \dots, \xi_n))\, \mathrm{d}\xi_1 \dots \mathrm{d}\xi_n. \tag{5}$$

The technique used is an extension to n variables of one employed by Hardy and Littlewood themselves for sums in one variable.

(*b*) We define $\psi_0(\xi)$ for real ξ by

$$\psi_0(\xi) = \xi - [\xi] - \tfrac{1}{2}, \tag{6}$$

and have

$$\psi_0(\xi) = -\sum_{l=-\infty}^{\infty}{}' \frac{\mathrm{e}\,(l\xi)}{2\pi i l}, \tag{7}$$

where the accent indicates that $l = 0$ is omitted and that the terms l, $-l$ are taken together. We define $\psi_1(\xi)$, $\psi_2(\xi)$, ... by

$$\psi_r(\xi) = -\psi'_{r+1}(\xi), \quad \int_0^1 \psi_r(\xi)\, \mathrm{d}\xi = 0, \tag{8}$$

and have

$$\psi_r(\xi) = (-1)^{r+1} \sum_{l=-\infty}^{\infty}{}' \frac{e\,(l\xi)}{(2\pi i l)^{r+1}}. \tag{9}$$

Lemma 5·1. *Let A and B be non-integral real numbers satisfying $B > A$. Let $f^{(s)}(\xi)$ exist and be continuous for $A \leqslant \xi \leqslant B$. Then*

$$\sum_{A<x<B} f(x) = \int_A^B f(\xi)\, \mathrm{d}\xi - \sum_{h=0}^{s-1} [\psi_h(\xi) f^{(h)}(\xi)]_{\xi=A}^{\xi=B} + \int_A^B \psi_{s-1}(\xi) f^{(s)}(\xi)\, \mathrm{d}\xi. \tag{10}$$

Proof. Landau (1927, Satz 331), with a trifling change of notation.

Lemma 5·2. *Let q be a positive integer and let z be an integer. Then, with the hypotheses of lemma* 5·1, *we have*

$$\sum_{\substack{A<x<B \\ x \equiv z\, (\mathrm{mod}\, q)}} f(x) = q^{-1} \int_A^B f(\xi)\, \mathrm{d}\xi - \sum_{h=0}^{s-1} \left[q^h \psi_h \left(\frac{\xi - z}{q} \right) f^{(h)}(\xi) \right]_{\xi=A}^{\xi=B} + q^{s-1} \int_A^B \psi_{s-1} \left(\frac{\xi - z}{q} \right) f^{(s)}(\xi)\, \mathrm{d}\xi. \tag{11}$$

Proof. Put $f_0(\eta) = f(q\eta + z)$, $A_0 = (A-z)/q$, $B_0 = (B-z)/q$ and apply lemma 5·1 to the function $f_0(\eta)$ in the interval $A_0 \leqslant \eta \leqslant B_0$.

(*c*) We shall first approximate to the sum S defined by

$$S = \sum_{\substack{A_j < x_j < B_j \\ \mathbf{x} \equiv \mathbf{z}\, (\mathrm{mod}\, q)}} \mathrm{e}\,(\beta C(\mathbf{x})), \tag{12}$$

and we shall do so by making n successive applications of lemma 5·2, one for each variable of summation. It will be convenient to denote by ∂_j the operation of partial differentiation with respect to ξ_j, for $j = 1, \dots, n$. All the applications will be made with the same value of s; we choose s to be the least positive integer such that

$$s\delta > n. \tag{13}$$

Thus s is fixed and independent of P.

Lemma 5·3. *Suppose that $0 \leqslant h_j \leqslant s$ $(j = 1, \dots, n)$. Then, for $A_j \leqslant \xi_j \leqslant B_j$, we have*

$$|\, \partial_1^{h_1} \dots \partial_n^{h_n}\, \mathrm{e}\,(\beta C(\xi))\, | \ll (q^{-1} P^{-\delta})^{h_1 + \dots + h_n}. \tag{14}$$

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Proof. Let $h = h_1 + \ldots + h_n$. It is easily proved by induction that the explicit expression for

$$\partial_1^{h_1} \ldots \partial_n^{h_n} \,\mathrm{e}\,(\beta C(\boldsymbol{\xi}))$$

is of the form
$$\{\beta^h \Phi_{2h} + \beta^{h-1}\Phi_{2h-3} + \beta^{h-2}\Phi_{2h-6} + \ldots\}\,\mathrm{e}\,(\beta C(\boldsymbol{\xi})),$$

where Φ_r denotes a homogeneous polynomial of degree r in $\xi_1, \ldots, \xi_n$, and the series terminates with the last such polynomial of non-negative degree. Since $|\Phi_r| \ll P^r$, the above expression is majorized by

$$(q^{-1}P^{-2-\delta})^h P^{2h} + (q^{-1}P^{-2-\delta})^{h-1} P^{2h-3} + \ldots \ll (q^{-1}P^{-\delta})^h,$$

since $q \leqslant P^{1-\delta}$. Hence the result.

LEMMA 5·4. *The sum* (12) *differs from*

$$q^{-n}\int_{A_1}^{B_1} \ldots \int_{A_n}^{B_n} \mathrm{e}\,(\beta C(\xi_1, \ldots, \xi_n))\, \mathrm{d}\xi_1 \ldots \mathrm{d}\xi_n \tag{15}$$

by the sum of a bounded number of expressions of the kind typified below, together with an error term which is

$$O(q^{-n}).$$

The typical expression is

$$q^{r-n+h_1+\ldots+h_r}\,\psi_{h_1}\!\left(\frac{B_1 - z_1}{q}\right) \ldots \psi_{h_r}\!\left(\frac{B_r - z_r}{q}\right)$$
$$\times \int_{A_{r+1}}^{B_{r+1}} \ldots \int_{A_n}^{B_n} \partial_1^{h_1} \ldots \partial_r^{h_r}\, \mathrm{e}\,(\beta C(B_1, \ldots, B_r, \xi_{r+1}, \ldots, \xi_n))\, \mathrm{d}\xi_{r+1} \ldots \mathrm{d}\xi_n, \tag{16}$$

where $1 \leqslant r \leqslant n$, *and* $h_1, \ldots, h_r$ *can each take any of the values* $0, 1, \ldots, s-1$, *and any or all of* $B_1, \ldots, B_r$ *may be replaced by the corresponding* A.

Proof. Lemma 5·2 expresses the operation of summation over x_j, subject to $x_j \equiv z_j \pmod{q}$, as equivalent to the sum of three other operations. The first is integration over ξ_j, with a factor q^{-1} prefixed. The second is the formation of a linear combination of the values of the summand and its derivatives at the end points A_j, B_j. The third is an integration involving the sth derivative of the summand.

Consider first those expressions which arise from various choices (in each of the n applications of the lemma) made from the first two operations only. If integration is selected every time, we get the principal expression (15). If integration is selected $n-r$ times, say for the variables $\xi_{r+1}, \ldots, \xi_n$, and if end-point values are selected in the other r applications, we get an expression of the form (16), where r has one of the values $1, \ldots, n$.

Consider next those expressions which arise when the third operation is selected at least once. Take for simplicity the case in which the third operation is selected for ξ_1 and the first operation for $\xi_2, \ldots, \xi_n$, giving

$$q^{s-n}\int_{A_1}^{B_1} \ldots \int_{A_n}^{B_n} \psi_{s-1}\!\left(\frac{\xi_1 - z_1}{q}\right) \partial_1^s\, \mathrm{e}\,(\beta C(\xi_1, \ldots, \xi_n))\, \mathrm{d}\xi_1 \ldots \mathrm{d}\xi_n.$$

By lemma 5·3 and the fact that ψ_{s-1} is bounded, this has absolute value

$$\ll q^{s-n} P^n (q^{-1}P^{-\delta})^s \ll q^{-n}.$$

Take next the case in which the third operation is selected for ξ_1 and the second operation for $\xi_2, \ldots, \xi_n$, giving an expression of the form

$$q^{s-1+h_2+\ldots+h_n}\int_{A_1}^{B_1}\psi_{s-1}\left(\frac{\xi_1-z_1}{q}\right)\psi_{h_2}\left(\frac{B_2-z_2}{q}\right)\ldots\psi_{h_n}\left(\frac{B_n-z_n}{q}\right)$$
$$\times\,\partial_1^{s_1}\,\partial_2^{h_2}\ldots\partial_n^{h_n}\,\mathrm{e}\,(\beta C(\xi_1, B_2, \ldots, B_n))\,\mathrm{d}\xi_1.$$

This is majorized by

$$q^{s-1+h_2+\ldots+h_n}P(q^{-1}P^{-\delta})^{s+h_2+\ldots+h_n} \ll q^{-1}P^{1-s\delta} \ll q^{-1}P^{1-n} \ll q^{-n}.$$

It will now be seen, from the structure of these estimates, that the other possible cases are no less favourable. In fact, if the first operation is chosen μ times, the third operation ν times $(\nu \geqslant 1)$ and the second operation $n-\mu-\nu$ times, the estimate for the resulting expression is

$$q^{-\mu}P^{\mu}q^{(s-1)\nu}P^{\nu}(q^{-1}P^{-\delta})^{s\nu} = q^{-\mu-\nu}P^{\mu+\nu-\delta s\nu} \ll q^{-\mu-\nu}P^{\mu+\nu-n} \ll q^{-n}.$$

This proves the result.

(*d*) For $1 \leqslant r \leqslant n$, and any integers $h_1, \ldots, h_r$, and any $B_1, \ldots, B_r$ which differ by $\frac{1}{2}$ from integers, we define

$$T(h_1, \ldots, h_r) = \sum_{z_1=1}^{q}\ldots\sum_{z_n=1}^{q}\mathrm{e}\left(\frac{a}{q}C(\mathbf{z})\right)\psi_{h_1}\left(\frac{B_1-z_1}{q}\right)\ldots\psi_{h_r}\left(\frac{B_r-z_r}{q}\right). \tag{17}$$

Lemma 5·5. *Suppose that*

$$|\,T(h_1, \ldots, h_r)\,| \leqslant T \tag{18}$$

for all $h_1, \ldots, h_r$ *satisfying* $0 \leqslant h_j \leqslant s$ $(1 \leqslant j \leqslant r)$. *Then, if* (1) *and* (2) *hold, and* $\beta = \alpha - a/q$, *we have*

$$S(\alpha) = q^{-n}S_{a,q}I(\beta) + O(T(P/q)^{n-1}) + O(1). \tag{19}$$

Proof. We have

$$S(\alpha) = \sum_{z_1=1}^{q}\ldots\sum_{z_n=1}^{q}\mathrm{e}\left(\frac{a}{q}C(\mathbf{z})\right)\sum_{\substack{A_1<x_1<B_1\\ x_1\equiv z_1\,(\mathrm{mod}\ q)}}\ldots\sum_{\substack{A_n<x_n<B_n\\ x_n\equiv z_n\,(\mathrm{mod}\ q)}}\mathrm{e}\,(\beta C(\mathbf{x})).$$

We substitute for the inner sum from lemma 5·4. The principal term gives $q^{-n}S_{a,q}I(\beta)$. The error term $O(q^{-n})$ in lemma 5·4 gives $O(1)$. It remains to consider the result of substituting the typical expression (16). This result is

$$q^{r-n}T(h_1, \ldots, h_r)\,I_{n-r},$$

where
$$I_{n-r} = q^{h_1+\ldots+h_r}\int_{A_{r+1}}^{B_{r+1}}\ldots\int_{A_n}^{B_n}\partial_1^{h_1}\ldots\partial_r^{h_r}\,\mathrm{e}\,(\beta C(\boldsymbol{\xi}))\,\mathrm{d}\xi_{r+1}\ldots\mathrm{d}\xi_n.$$

By lemma 5·3, we have

$$|\,I_{n-r}\,| \ll q^{h_1+\ldots+h_r}P^{n-r}(q^{-1}P^{-\delta})^{h_1+\ldots+h_r}$$
$$\ll P^{n-r}.$$

Hence the resulting contribution to $S(\alpha)$ is

$$\ll T(P/q)^{n-r} \ll T(P/q)^{n-1},$$

since $r \geqslant 1$. This completes the proof.

Lemma 5·6. *Let* $C(\mathbf{x})$ *be a fixed cubic form in n variables, with integral coefficients, which does not represent zero. Suppose that* $n \geqslant 2$, *and that* κ *is a fixed number satisfying*

$$0 < \kappa < \tfrac{1}{8}n. \tag{20}$$

Suppose that $C(\mathbf{x})$ *does not split with remainder* m, *where* m *is the least integer satisfying*

$$m \geqslant n-4\kappa. \tag{21}$$

Then, subject to (1) *and* (2), *we have*

$$S(\alpha) = q^{-n}S_{a,q}I(\beta)+O(P^{n-1}q^{1-\kappa}(\log q)^n). \tag{22}$$

Proof. By lemma 5·5 it suffices to prove that

$$|\,T(h_1, \ldots, h_r)\,| \ll q^{n-\kappa}(\log q)^n. \tag{23}$$

By (9) we have

$$\psi_h(\xi) = (-1)^{h+1}\sum_{l=-q^n}^{q^n}{}' \frac{\mathrm{e}\,(l\xi)}{(2\pi i l)^{h+1}}+R,$$

where

$$|\,R\,| \ll \frac{1}{(q^n)^{h+1}}\frac{1}{\|\,\xi\,\|} \ll \frac{1}{q^n\,\|\,\xi\,\|}.$$

Hence

$$\psi_h\left(\frac{B-z}{q}\right) = (-1)^{h+1}\sum_{l=-q^n}^{q^n}{}' (2\pi i l)^{-h-1}\,\mathrm{e}\left(\frac{l(B-z)}{q}\right)+R', \tag{24}$$

where

$$|\,R'\,| \ll q^{1-n},$$

since $B-z$ is half an odd integer.

Substitution in the definition of $T(h_1, \ldots, h_r)$ in (17) gives for the main term (apart from a constant factor)

$$\sum_{l_1=-q^n}^{q^n}{}' \cdots \sum_{l_r=-q^n}^{q^n}{}' l_1^{-h_1-1}\ldots l_r^{-h_r-1}\sum_{\mathbf{z}} \mathrm{e}\left(\frac{a}{q}C(\mathbf{z})+\frac{l_1(B_1-z_1)}{q}+\ldots+\frac{l_r(B_r-z_r)}{q}\right),$$

where the summation for $\mathbf{z}$ is over complete sets of residues (mod q) for each of $z_1, \ldots, z_n$. The inner sum above is

$$S_{a,q}(-l_1, \ldots, -l_r, 0, \ldots, 0),$$

apart from a factor of absolute value 1.

It follows from lemma 4·3, in view of our hypotheses, that the last sum has absolute value $\ll q^{n-\kappa}$. Hence the above principal part has absolute value

$$\ll q^{n-\kappa}\sum_{l_1=-q^n}^{q^n}{}' \cdots \sum_{l_r=-q^n}^{q^n}{}' |\,l_1 \ldots l_r\,|^{-1}$$
$$\ll q^{n-\kappa}(\log q)^r \ll q^{n-\kappa}(\log q)^n.$$

As regards the remainder after multiplying together r formulae of the type (24), if we take the error term in one formula and the principal term in the other $r-1$ formulae, we obtain a result which is

$$\ll \sum_{l_2=-q^n}^{q^n}{}' \cdots \sum_{l_r=-q^n}^{q^n}{}' |\,l_2 \ldots l_r\,|^{-1}q^{1-n}\sum_{\mathbf{z}} 1$$
$$\ll q(\log q)^{r-1} \ll q^{n-\kappa}(\log q)^n,$$

since $n \geqslant 2$. The result of taking more than one error term is plainly better. This proves (23), and hence the result.

6. The choice of intervals

(*a*) So far we have imposed two conditions on the box $P\mathscr{B}$ occurring in the definition of $S(\alpha)$ in (2) of § 4. The first is that the box $\mathscr{B}$ itself shall satisfy (3) of § 3, but this is unimportant. The second is that the box $P\mathscr{B}$ shall be of the form given in (3) of § 5, where $A_j-\frac{1}{2}$ and $B_j-\frac{1}{2}$ are integers.

There is a more important condition which has to be satisfied by the box $\mathscr{B}$; we have to ensure that the density of the real solutions of $C(\mathbf{x}) = 0$ with $\mathbf{x}$ in $\mathscr{B}$ is not abnormally small. The aim of this section is to satisfy such a requirement by taking $\mathscr{B}$ so that it contains a suitable non-singular real solution of $C(\mathbf{x}) = 0$.

We recall that $I(\beta)$ was defined in (5) of § 5, and the definition can be restated as

$$I(\beta) = P^n \int_{\mathscr{B}} \mathrm{e}\,(\beta P^3 C(\boldsymbol{\xi}))\, \mathrm{d}\boldsymbol{\xi}. \tag{1}$$

Thus, for $\mu > 0$,
$$\int_{-\mu}^{\mu} I(\beta)\, \mathrm{d}\beta = P^n \int_{\mathscr{B}} \frac{\sin 2\pi\mu P^3 C(\boldsymbol{\xi})}{\pi P^3 C(\boldsymbol{\xi})}\, \mathrm{d}\boldsymbol{\xi},$$

whence
$$\int_{-\mu}^{\mu} I(\beta)\, \mathrm{d}\beta = P^{n-3} J(P^3\mu), \tag{2}$$

where
$$J(\phi) = \int_{\mathscr{B}} \frac{\sin 2\pi\phi\, C(\boldsymbol{\xi})}{\pi C(\boldsymbol{\xi})}\, \mathrm{d}\boldsymbol{\xi}. \tag{3}$$

The precise requirement which we wish to satisfy is that

$$\lim_{\phi \to \infty} J(\phi)$$

shall exist and have a positive value.

(*b*) Lemma 6·1. *There exist real numbers $\xi_1, \ldots, \xi_n$ such that*

$$C(\xi_1, \ldots, \xi_n) = 0, \tag{4}$$

$$\xi_1 \neq 0, \ldots, \xi_n \neq 0, \tag{5}$$

$$\partial C / \partial \xi_1 \neq 0. \tag{6}$$

Proof. For any $\xi_2, \ldots, \xi_n$ the equation (4) is of the form

$$c_{111}\xi_1^3 + 3F\xi_1^2 + 3G\xi_1 + H = 0, \tag{7}$$

where F, G, H are respectively linear, quadratic and cubic forms in $\xi_2, \ldots, \xi_n$ with integral coefficients. Since we suppose that $C(x_1, \ldots, x_n)$ does not represent zero, we have $c_{111} \neq 0$ and $H = C(0, \xi_2, \ldots, \xi_n)$ is not identically zero.

Since (7) is a cubic equation in ξ_1, it has a real solution for any $\xi_2, \ldots, \xi_n$. The condition $\xi_1 \neq 0$ is satisfied if $H \neq 0$. The condition $\partial C / \partial \xi_1 \neq 0$ is satisfied provided $D \neq 0$, where D denotes the discriminant in ξ_1 of the equation (7). We note that D cannot be identically zero in $\xi_2, \ldots, \xi_n$, for if so there would exist an identity of the form

$$C(\xi_1, \ldots, \xi_n) = c_{111}(\xi_1 - \psi)^2 (\xi_1 - \chi),$$

where ψ, χ are rational functions of $\xi_2, \ldots, \xi_n$ with rational coefficients, and then the equation $C(x_1, \ldots, x_n) = 0$ would have a non-zero rational solution, contrary to hypothesis.

Thus it suffices to choose any real $\xi_2, \ldots, \xi_n$ which satisfy $\xi_2 \neq 0, \ldots, \xi_n \neq 0$ and

$$H(\xi_2, \ldots, \xi_n) \neq 0, \quad D(\xi_2, \ldots, \xi_n) \neq 0.$$

This is obviously possible.

Lemma 6·2. *Let $\xi_1^*, \ldots, \xi_n^*$ be real numbers satisfying the conditions* (4), (5), (6). *If ρ is a sufficiently small positive number, and if $\mathscr{B}$ is a box of the form*

$$\xi_j' < \xi_j < \xi_j'' \quad (1 \leqslant j \leqslant n), \tag{8}$$

where $$\xi_j^* - \rho < \xi_j' < \xi_j^* < \xi_j'' < \xi_j^* + \rho, \tag{9}$$

then the function $J(\phi)$ defined by (3) *satisfies*

$$\lim_{\phi \to \infty} J(\phi) = J_0 > 0. \tag{10}$$

Proof. We can write

$$C(\boldsymbol{\xi}^* + \boldsymbol{\eta}) = \eta_1 + c_2\eta_2 + \ldots + c_n\eta_n + P_2(\boldsymbol{\eta}) + P_3(\boldsymbol{\eta}),$$

where we have taken the coefficient of η_1 to be 1, as we can do without loss of generality in view of (6). If ρ is sufficiently small, and $|\boldsymbol{\eta}| < \rho$, the equation

$$C(\xi_1^* + \eta_1, \ldots, \xi_n^* + \eta_n) = \zeta$$

implies $|\zeta| < \sigma$, where σ is small with ρ. Further, this equation will have a unique solution for η_1 in terms of $\eta_2, \ldots, \eta_n$ and ζ, the solution being of the form

$$\eta_1 = \zeta - c_2\eta_2 - \ldots - c_n\eta_n + \Phi(\zeta, \eta_2, \ldots, \eta_n),$$

where Φ is a multiple power series in which all terms are of degree 2 at least. This power series is absolutely convergent for

$$|\zeta| < \sigma, \quad |\eta_2| < \rho, \ldots, \quad |\eta_n| < \rho. \tag{11}$$

We have $$\frac{\partial \eta_1}{\partial \zeta} = 1 + \Phi_1(\zeta, \eta_2, \ldots, \eta_n),$$

where Φ_1 is a multiple power series with no constant term. We can suppose that $|\Phi_1| < \frac{1}{2}$ in the region (11).

By (3), $$J(\phi) = \int_{\mathscr{B}'} \frac{\sin 2\pi\phi C(\boldsymbol{\xi}^* + \boldsymbol{\eta})}{\pi C(\boldsymbol{\xi}^* + \boldsymbol{\eta})} \, d\boldsymbol{\eta},$$

where $\mathscr{B}'$ is a box of the form $-\rho_j' < \eta_j < \rho_j''$ and $0 < \rho_j' < \rho$, $0 < \rho_j'' < \rho$. Making a change of variable from η_1 to ζ, we obtain

$$J(\phi) = \int_{\mathscr{R}} \frac{\sin 2\pi\phi\zeta}{\pi\zeta} \{1 + \Phi_1(\zeta, \eta_2, \ldots, \eta_n)\} \, d\zeta \, d\eta_2 \ldots d\eta_n,$$

where $\mathscr{R}$ is the n-dimensional region defined by

$$|\zeta| < \sigma, \quad -\rho_j' < \eta_j < \rho_j'' \quad (2 \leqslant j \leqslant n), \quad -\rho_1' < \zeta - c_2\eta_2 - \ldots - c_n\eta_n + \Phi(\zeta, \eta_2, \ldots, \eta_n) < \rho_1''.$$

Let $\mathscr{R}(\zeta)$ denote the $(n-1)$-dimensional region in the space of $\eta_2, \ldots, \eta_n$ defined by these inequalities for a particular value of ζ, and let

$$V(\zeta) = \int_{\mathscr{R}(\zeta)} \{1 + \Phi_1(\zeta, \eta_2, \ldots, \eta_n)\} \, d\eta_2 \ldots d\eta_n. \tag{12}$$

Then $$J(\phi) = \int_{-\sigma}^{\sigma} \frac{\sin 2\pi\phi\zeta}{\pi\zeta} V(\zeta) \, d\zeta.$$

It is plain that, if ρ is sufficiently small, the integral $V(\zeta)$ will be continuous and of bounded variation for $-\sigma<\zeta<\sigma$. Hence, by a classical result, the limit of $J(\phi)$ as $\phi\to\infty$ exists and has the value $V(0)$. By (12), $V(0)$ is an integral over the $(n-1)$-dimensional region defined by

$$-\rho_j'<\eta_j<\rho_j'' \quad (2\leqslant j\leqslant n), \quad -\rho_1'<-c_2\eta_2-\ldots-c_n\eta_n+\Phi(0,\eta_2,\ldots,\eta_n)<\rho_1'',$$

and the integrand is greater than $\frac{1}{2}$. Since the region includes some $n-1$-dimensional cube round the origin, we have $V(0)>0$, and this proves the result.

(c) We can obviously take $\xi_1', \xi_1'', \ldots, \xi_n', \xi_n''$ to satisfy (9) and to be of the form

$$\xi_j'=\frac{a_j}{2N}, \quad \xi_j''=\frac{b_j}{2N},$$

where N is a positive integer and a_j, b_j are odd integers. The condition (3) of § 3 is satisfied if $x_j'=\xi_j'$, $x_j''=\xi_j''$, in view of (9), since ρ is small. The box $P\mathscr{B}$ will be of the form postulated in (3) of § 5 provided $P\xi_j'$, $P\xi_j''$ differ by $\frac{1}{2}$ from integers. This will be so if P is an odd multiple of N, and henceforward we impose this restriction on P.

7. The singular series

(a) Let $C\,(x_1,\ldots,x_n)$ be a cubic form with integral coefficients which is non-degenerate in the sense of § 2. Let

$$S_{a,q}=\sum_{\mathbf{x}}\mathrm{e}\left(\frac{a}{q}C(\mathbf{x})\right) \tag{1}$$

as in (4) of § 4, the summation being over n complete sets of residues $(\operatorname{mod} q)$. The *singular series* which is relevant to the problem of representing zero by $C(\mathbf{x})$ is

$$\mathfrak{S}=\sum_{q=1}^{\infty}\sum_{\substack{a=1\\(a,q)=1}}^{q}q^{-n}S_{a,q}. \tag{2}$$

We shall suppose throughout this section that

$$|S_{a,q}|\ll q^{n-2-\delta} \tag{3}$$

for some fixed positive δ; this plainly ensures the absolute convergence of the series (2). In the applications later, the verification that (3) holds will be based partly on lemma 4·3 and partly on classical results for sums in one variable.

The following results are well known in the classical case when $C(\mathbf{x})$ is a sum of cubes, and the proofs given then, for example in Vinogradov (1954, chapter II), remain valid under the present more general circumstances.

Lemma 7·1. *We have*

$$\mathfrak{S}=\prod_{p}\chi(p),$$

where the product is extended over all primes p and

$$\chi(p)=1+\sum_{l=1}^{\infty}\sum_{\substack{a=1\\(a,p)=1}}^{p^l}p^{-ln}S_{a,p^l}.$$

Lemma 7·2. *We have*

$$\chi(p)=\lim_{\nu\to\infty}p^{-\nu(n-1)}M(p^\nu),$$

where $M(p^\nu)$ denotes the number of solutions of the congruence

$$C(x_1, \ldots, x_n) \equiv 0 \pmod{p^\nu}$$

with

$$0 \leqslant x_j < p^\nu \quad (j = 1, \ldots, n).$$

(*b*) We can now prove the one result that will be needed concerning the singular series.

Lemma 7·3. *Suppose that $n \geqslant 10$ and that $C(x_1, \ldots, x_n)$ is non-degenerate. Then*

$$\mathfrak{S} > 0.$$

Proof. By lemma 2·8, the form $C(\mathbf{x})$ has the property $\mathscr{A}(p^l)$ for some l. By lemma 2·3, this implies that

$$M(p^{2l-1+\nu}) \geqslant p^{(n-1)\nu}$$

for every positive integer ν. Hence

$$M(p^\nu) \geqslant A_p p^{(n-1)\nu}$$

for all sufficiently large ν, where A_p is a positive number depending on p. Thus, by lemma 7·2 above, we have

$$\chi(p) \geqslant A_p.$$

Furthermore, by lemma 7·1 above and the hypothesis (3), we have

$$|\chi(p) - 1| \ll \sum_{l=1}^{\infty} p^l p^{-ln} p^{l(n-2-\delta)} \ll p^{-1-\delta}.$$

Hence

$$\chi(p) > 1 - Bp^{-1-\delta},$$

where B is independent of p.

Choosing p_0 suitably, we obtain

$$\mathfrak{S} \geqslant \{\prod_{p \leqslant p_0} A_p\}\{\prod_{p > p_0} (1 - Bp^{-1-\delta})\} > 0.$$

8. Synthesis

(*a*) Let $C(x_1, \ldots, x_n)$ be a cubic form with integral coefficients, and suppose that $n \geqslant 32$. As explained in § 1, the proof is effected in a number of stages. We prove first, under the heading $[n; 0]$, that if $C(\mathbf{x})$ does not represent zero then it splits with remainder $n-8$, that is, it represents a form of the type

$$C'(x_1, \ldots, x_{n-8}) + d_1 y_1^3.$$

Put $n_1 = n - 8$. Under the heading $[n_1; 1]$ we consider the last form *de novo*, and prove that if it does not represent zero then C' splits with remainder $n_1 - 6$. The process continues under further headings until finally it suffices to prove, as we do under the heading $[n_7; 7]$, that a form of the type

$$d_1 y_1^3 + \ldots + d_8 y_8^3$$

always represents zero.

The proof at each stage is based on the Hardy–Littlewood method, and the dissection of an interval of integration into 'major arcs' and 'minor arcs' is made in the same way throughout. For any integers a, q satisfying

$$1 \leqslant q \leqslant P^{1-2\delta}, \quad 1 \leqslant a \leqslant q, \quad (a, q) = 1, \tag{1}$$

we define the interval $\mathfrak{M}_{a,q}$ to consist of all real α satisfying

$$|\alpha - a/q| < q^{-1} P^{-2-\delta}. \tag{2}$$

It is almost immediate that two intervals, corresponding to different pairs a, q satisfying (1), do not overlap. Further, all the intervals $\mathfrak{M}_{a,q}$ are contained in an interval I of length 1, for example the interval from $P^{-2-\delta}$ to $1+P^{-2-\delta}$. We denote the remainder of the interval I, after the removal of all the intervals $\mathfrak{M}_{a,q}$, by $\mathfrak{m}$.

(b) *The case* $[n; 0]$

We choose a box $\mathscr{B}$ as in § 6 (*c*) and define $S(\alpha)$ by (2) of § 4. Then the number $\mathscr{N}(P)$ of integer points $\mathbf{x}$ in the box $P\mathscr{B}$ which satisfy $C(\mathbf{x}) = 0$ is given by

$$\mathscr{N}(P) = \int_I S(\alpha)\,\mathrm{d}\alpha,$$

and dissection of the interval of integration gives

$$\mathscr{N}(P) = \sum_q \sum_a \int_{\mathfrak{M}_{a,q}} S(\alpha)\,\mathrm{d}\alpha + \int_{\mathfrak{m}} S(\alpha)\,\mathrm{d}\alpha, \tag{3}$$

where the summation is over a, q satisfying (1).

We suppose that $C(\mathbf{x})$ does not split with remainder $n-8$, since this will be the case considered under the next heading.

To estimate the last integral in (3), we appeal to lemma 4·2. We use various values of θ in the range

$$\tfrac{1}{2}(1-3\delta) \leqslant \theta \leqslant \tfrac{3}{4}, \tag{4}$$

and for each θ we define κ by

$$\kappa = \tfrac{9}{2}\theta(1-\delta). \tag{5}$$

The condition $4\kappa < n\theta$ is equivalent to $n \geqslant 18$, and is satisfied. The inequality (9) of § 4 becomes $m \geqslant n-9(1-\delta)$, i.e. $m \geqslant n-8$, and the condition that $C(\mathbf{x})$ does not split with remainder m is satisfied. It follows from lemma 4·2 that, for any real α, either

$$|\,S(\alpha)\,| < P^{n-\frac{9}{2}\theta(1-\delta)} \tag{6}$$

or there is an approximation a'/q' to α which satisfies

$$1 \leqslant q' < P^{2\theta+\delta}, \quad |\,\alpha - a'/q'\,| < q'^{-1}P^{-3+2\theta+\delta}. \tag{7}$$

If α is in $\mathfrak{m}$, the latter alternative cannot arise when $\theta = \frac{1}{2}(1-3\delta)$, for then a', q' would satisfy (1) and (2), and α would be in $\mathfrak{M}_{a',q'}$. Hence (6) holds for all α in $\mathfrak{m}$ when $\theta = \frac{1}{2}(1-3\delta)$. For other values of θ in the range (4), the existence of a', q' satisfying (7) restricts α to a set $\mathscr{E}(\theta)$ whose measure $|\,\mathscr{E}(\theta)\,|$ satisfies

$$|\,\mathscr{E}(\theta)\,| < \sum_{q'} \sum_{a'} 2q'^{-1}P^{-3+2\theta+\delta} \ll P^{-3+4\theta+2\delta}. \tag{8}$$

This gives an estimate for the measure of those α for which (6) fails to hold.

We apply these results with a decreasing sequence $\theta_1, \theta_2, \ldots, \theta_h$ of values of θ, where

$$\theta_1 = \tfrac{3}{4}, \quad \theta_h = \tfrac{1}{2}(1-3\delta). \tag{9}$$

The contribution to $\int |\,S(\alpha)\,|\,\mathrm{d}\alpha$ made by those α which satisfy (6) when $\theta = \theta_1 = \frac{3}{4}$ is

$$\ll P^{n-\frac{27}{8}(1-\delta)} \ll P^{n-3-\delta}.$$

Consider next the contribution made by those α which satisfy (6) when $\theta = \theta_{g+1}$ but not when $\theta = \theta_g$, where $g = 1, 2, \ldots, h-1$. These α lie in $\mathscr{E}(\theta_g)$, so their contribution is

$$\ll P^{n-\frac{9}{2}\theta_{g+1}(1-\delta)}\,|\,\mathscr{E}(\theta_g)\,| \ll P^{n-3-\frac{9}{2}\theta_{g+1}+4\theta_g+6\delta}.$$

The exponent is less than $n-3-\delta$ provided θ_{g+1}/θ_g exceeds 8/9 by a fixed amount. Plainly we can find values for $\theta_1, \ldots, \theta_h$ which satisfy this condition. Finally, as remarked earlier, every α in $\mathfrak{m}$ satisfies (6) when $\theta = \theta_h$. Hence we have

$$\int_{\mathfrak{m}} |S(\alpha)|\, d\alpha \ll P^{n-3-\delta}. \tag{10}$$

In each interval $\mathfrak{M}_{a,q}$ we can approximate to $S(\alpha)$ by lemma 5·6, in which we can take $\kappa = \frac{9}{4}-\delta$, since then $C(\mathbf{x})$ does not split with remainder m satisfying (21) of § 5. Thus

$$S(\alpha) = q^{-n}S_{a,q}I(\beta) + O(P^{n-1}q^{-\frac{5}{4}+2\delta})$$

for α in $\mathfrak{M}_{a,q}$, where $\beta = \alpha - a/q$. The error term here, when integrated over (2) and summed over (1), gives an amount

$$\ll \sum_q \sum_a P^{n-1}q^{-\frac{5}{4}+2\delta}q^{-1}P^{-2-\delta} \ll P^{n-3-\delta}.$$

Hence
$$\sum_q \sum_a \int_{\mathfrak{M}_{a,q}} S(\alpha)\, \mathrm{d}\alpha = \sum_q \sum_a q^{-n}S_{a,q}\int I(\beta)\, \mathrm{d}\beta + O(P^{n-3-\delta}), \tag{11}$$

where the integration is over $|\beta| < q^{-1}P^{-2-\delta}$ and the summation is over a, q satisfying (1).

In the notation of (2) and (3) of § 6, the integral with respect to β is

$$P^{n-3}\, J(P^3q^{-1}P^{-2-\delta}).$$

Since $q^{-1}P^{1-\delta} \geq P^{\delta}$ by (1), it follows from lemma 6·2 and the choice of the box $\mathscr{B}$ in § 6 (c) that

$$|J(q^{-1}P^{1-\delta}) - J_0| < \epsilon(P),$$

where J_0 is a positive number independent of P and where $\epsilon(P)$ is independent of q and $\epsilon(P) \to 0$ as $P \to \infty$. Hence (11) implies

$$\left| \sum_q \sum_a \int_{\mathfrak{M}_{a,q}} S(\alpha)\, \mathrm{d}\alpha - P^{n-3}J_0 \sum_q \sum_a q^{-n}S_{a,q} \right| \ll \epsilon(P)\, P^{n-3} \sum_q \sum_a q^{-n}\, |S_{a,q}| + P^{n-3-\delta}, \tag{12}$$

where all the summations are over a, q satisfying (1).

By lemma 4·3, with $\kappa = \frac{9}{4}-\delta$, we have

$$|S_{a,q}| \ll q^{n-\frac{9}{4}+\delta}. \tag{13}$$

Thus the hypothesis (3) of § 7 is satisfied, and the 'singular series'

$$\mathfrak{S} = \sum_{q=1}^{\infty} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} q^{-n}S_{a,q} \tag{14}$$

is absolutely convergent and has a positive value. Using the estimate (13) in (12), we obtain

$$\left| \sum_q \sum_a \int_{\mathfrak{M}_{a,q}} S(\alpha)\, \mathrm{d}\alpha - P^{n-3}\, J_0\mathfrak{S} \right| \ll P^{n-3}(\epsilon(P) + P^{-\delta}).$$

Combining this with (10) and (3), we derive the (conditional) asymptotic formula

$$\mathscr{N}(P) \sim P^{n-3}J_0\mathfrak{S} \tag{15}$$

as $P \to \infty$. (P is restricted as stated in § 6 (c), but this is unimportant.) Plainly $\mathscr{N}(P) > 0$ for large P, and consequently $C(\mathbf{x})$ represents zero.

We can now pass to the next case, with $n_1 = n-8$.

(*c*) *The case* $[n_1; 1]$

We are now concerned with a cubic form in n_1+1 variables of the type

$$C(x_1, \ldots, x_{n_1}) + d_1 y_1^3, \tag{16}$$

though of course C here is not the original cubic form in n variables. We have $n_1 = n-8 \geqslant 24$.

Our first task is to choose a suitable box for the variables. This is most conveniently done by regarding the above form as

$$C^+(x_1, \ldots, x_{n_1+1}),$$

with $x_{n_1+1} = y_1$, and choosing a corresponding box $\mathscr{B}^+$ in n_1+1 dimensions in accordance with § 6 (*c*). Such a box, after magnification by P, comprises a box $P\mathscr{B}$ in n_1 dimensions for $x_1, \ldots, x_{n_1}$ and an interval for y_1 of the form

$$\lambda P < y_1 < \mu P, \tag{17}$$

where λ and μ are constants and $0<\lambda<\mu$ (or possibly $\lambda<\mu<0$, which comes to the same).

We define

$$T_1(\alpha) = \sum_{\lambda P<y_1<\mu P} \mathrm{e}\,(\alpha d_1 y_1^3), \tag{18}$$

and we define $S(\alpha)$ by (2) of § 4 with n_1 in place of n. The number $\mathscr{N}(P)$ of solutions of $C^+(\mathbf{x}) = 0$ with $x_1, \ldots, x_{n_1}$ in $P\mathscr{B}$ and y_1 in (17) is given by

$$\mathscr{N}(P) = \int_I S(\alpha)\, T_1(\alpha)\, \mathrm{d}\alpha. \tag{19}$$

We dissect the integral as in (3).

It will be necessary to quote some results concerning the sum $T_1(\alpha)$ from the literature of Waring's Problem. They are usually stated there with $d_1 = 1$ and with some other interval in place of (17), but they remain valid under the present conditions. It should be noted that d_1 is a constant in the present context.

We suppose that $C(x_1, \ldots, x_{n_1})$ does not split with remainder n_1-6, since this will be the next case.

To estimate the contribution of $\mathfrak{m}$ to the integral (19), we appeal first to lemma 4·2, as in the preceding case. For each θ in the range (4), we define κ by

$$\kappa = \tfrac{7}{2}\theta(1-\delta).$$

The condition $4\kappa < n_1\theta$ is equivalent to $n_1 \geqslant 14$, and is satisfied. The inequality (9) of § 4 becomes $m \geqslant n_1 - 7(1-\delta)$, or $m \geqslant n_1-6$, and the condition that $C(\mathbf{x})$ does not split with remainder m is satisfied. It follows from lemma 4·2 that either

$$|\,S(\alpha)\,| < P^{n_1-\frac{7}{2}\theta(1-\delta)}, \tag{20}$$

or there is an approximation a'/q' to α satisfying the same conditions (7) as in the preceding case, and so α lies in the same exceptional set $\mathscr{E}(\theta)$ as before. Again, all α in $\mathfrak{m}$ satisfy (20) when $\theta = \frac{1}{2}(1-3\delta)$.

We recall that

$$\int_{\mathfrak{m}} |\,T_1(\alpha)\,|^4\, \mathrm{d}\alpha \leqslant \int_0^1 |\,T_1(\alpha)\,|^4\, \mathrm{d}\alpha \ll P^{2+\epsilon} \tag{21}$$

for any fixed $\epsilon>0$. This is a well-known consequence of the fact that the number of representations of a positive integer u as a sum of two integral cubes is $O(u^{\epsilon})$. It follows from Hölder's inequality that

$$\int_{\mathscr{E}(\theta)} |\,T_1(\alpha)\,|\,\mathrm{d}\alpha \ll \left\{\int_{\mathfrak{m}} |\,T_1(\alpha)\,|^4\,\mathrm{d}\alpha\right\}^{\frac{1}{4}} \{|\,\mathscr{E}(\theta)\,|\}^{\frac{3}{4}} \ll P^{-\frac{7}{4}+3\theta+2\delta}, \tag{22}$$

by (8) and (21).

As in the previous case, we consider a decreasing sequence $\theta_1, \ldots, \theta_h$ of values of θ, beginning and ending as in (9). The contribution to (19) of those α which satisfy (20) when $\theta=\theta_1=\frac{3}{4}$ is

$$\ll P^{n_1-\frac{21}{8}(1-\delta)} \int_{\mathfrak{m}} |\,T_1(\alpha)\,|\,\mathrm{d}\alpha \ll P^{n_1-\frac{21}{8}+3\delta+\frac{1}{2}}, \tag{23}$$

by (21) and Hölder's inequality. The exponent is less than $n_1-2-\delta$, which suffices for our purpose.

The contribution to (19) of those α which satisfy (20) when $\theta=\theta_{g+1}$ but not when $\theta=\theta_g$ is

$$\ll P^{n_1-\frac{7}{2}\theta_{g+1}(1-\delta)-\frac{7}{4}+3\theta_g+2\delta},$$

on using (22). The exponent is less than

$$n_1-2+3(\theta_g-\theta_{g+1})-\tfrac{1}{2}(\theta_{g+1}-\tfrac{1}{2})+5\delta, \tag{24}$$

and this is less than $n-2-\delta$ provided that

$$\theta_g-\theta_{g+1}<\tfrac{1}{3}\delta \quad \text{and} \quad \theta_{g+1}\geqslant\tfrac{1}{2}+14\delta. \tag{25}$$

Thus we can choose a number of values of θ which is bounded in terms of δ in such a way as to cover the contributions to (19) from all α except for those which fail to satisfy (20) when $\theta=\frac{1}{2}+14\delta$. This argument is therefore not completely effective, and we go on to supplement it by a further consideration.

Let $\mathscr{F}$ denote the set of those α in $\mathfrak{m}$ which fail to satisfy (20) when $\theta=\frac{1}{2}+14\delta$. By lemma 4·2 there exist a', q' such that

$$1\leqslant q'<P^{1+29\delta}, \quad (a',q')=1, \tag{26}$$

$$|\,\alpha-a'/q'\,|<q'^{-1}P^{-2+29\delta}. \tag{27}$$

Lemma 5 of Davenport & Heilbronn (1937) states that

$$T_1(\alpha)=q'^{-1}T_{a'd_1,q'}I^{(1)}(\beta')+O\{q'^{\frac{2}{3}+\epsilon}(1+P^3\,|\,\beta'\,|)\}, \tag{28}$$

where $\beta'=\alpha-a'/q'$ and

$$T_{b',q'}=\sum_{y=1}^{q'} \mathrm{e}\,(b'y^3/q'), \tag{29}$$

$$I^{(1)}(\beta')=\int_{\lambda P}^{\mu P} \mathrm{e}\,(d_1\beta'\eta^3)\,\mathrm{d}\eta. \tag{30}$$

It is well known that $|\,T_{b',q'}\,|\ll q'^{\frac{2}{3}}$ if $(b',q')=1$, and this remains valid for $T_{a'd_1,q'}$ since the highest common factor of $a'd_1$ and q' is bounded. It follows from (28) that

$$|\,T_1(\alpha)\,|\ll(q')^{-\frac{1}{3}}P+q'^{\frac{2}{3}+\epsilon}(1+q'^{-1}P^{1+29\delta})\ll(q')^{-\frac{1}{3}+\epsilon}P^{1+29\delta} \tag{31}$$

for all α in $\mathscr{F}$.

This enables us to deal with the exceptional set $\mathscr{F}$. We have

$$\int_{\mathscr{F}} |\,T_1(\alpha)\,|\,\mathrm{d}\alpha \ll \sum_{q' < P^{1+29\delta}} q'(q')^{-\frac{1}{3}+\epsilon} P^{1+29\delta} q'^{-1} P^{-2+29\delta} \ll P^{-\frac{1}{3}+78\delta}. \tag{32}$$

Since (20) holds throughout $\mathfrak{m}$ when $\theta = \frac{1}{2}(1-3\delta)$, we have

$$\int_{\mathscr{F}} |\,S(\alpha)\,T_1(\alpha)\,|\,\mathrm{d}\alpha \ll P^{n_1-\frac{7}{4}(1-\delta)(1-3\delta)-\frac{1}{3}+78\delta} \ll P^{n_1-2-\delta}.$$

This completes the proof that

$$\int_{\mathfrak{m}} |\,S(\alpha)\,T_1(\alpha)\,|\,\mathrm{d}\alpha \ll P^{n_1-2-\delta}.$$

Turning now to the intervals $\mathfrak{M}_{a,q}$, we have the result of lemma 5·6 with $k = \frac{7}{4}-\delta$, namely

$$S(\alpha) = q^{-n_1} S_{a,q} I(\beta) + O(P^{n_1-1} q^{-\frac{3}{4}+2\delta}) \tag{33}$$

for α in $\mathfrak{M}_{a,q}$, where of course $S_{a,q}$ and $I(\beta)$ are constructed from the form $C(x_1, \ldots, x_{n_1})$ in (16). We require also a similar approximation for $T_1(\alpha)$. That given above in (28), which is valid unconditionally, is not sufficiently precise for the present purpose. But by lemma 7 of Davenport (1939) we have

$$T_1(\alpha) = q^{-1} T_{ad_1,q} I^{(1)}(\beta) + O(q^{\frac{2}{3}+\epsilon}) \tag{34}$$

on $\mathfrak{M}_{a,q}$, where $T_{ad_1,q}$ and $I^{(1)}(\beta)$ are as defined in (29) and (30).

On multiplying together the main terms in (33) and (34), we obtain a main term which is simply

$$q^{-(n_1+1)} S^+_{a,q} I^+(\beta),$$

where the superscript indicates that the sum and the integral are defined in terms of the form $C^+(x_1, \ldots, x_{n_1+1})$ in n_1+1 variables, with the appropriate box $P\mathscr{B}^+$ in the case of the integral. The error term in the product is easily estimated, since

$$|\,S_{a,q}\,| \ll q^{n_1-\frac{7}{4}+\delta} \tag{35}$$

by lemma 4·3 with $\kappa = \frac{7}{4}-\delta$, and

$$|\,T_{ad_1,q}\,| \ll q^{\frac{2}{3}}, \tag{36}$$

as noted earlier. It is found that the error term is $\ll P^{n_1} q^{-1-\frac{1}{12}+2\delta}$, and when this is integrated over $\mathfrak{M}_{a,q}$ and summed over (1) it gives a result $\ll P^{n_1-2-\delta}$.

Hence the contribution of the $\mathfrak{M}_{a,q}$ to the integral (19) can be taken to be

$$\sum_q \sum_a q^{-(n_1+1)} S^+_{a,q} \int I^+(\beta)\,\mathrm{d}\beta,$$

where the summation is over (1) and the integration over $|\,\beta\,| < q^{-1}P^{-2-\delta}$. We have

$$|\,S^+_{a,q}\,| = |\,S_{a,q}\,T_{ad_1,q}\,| \ll q^{(n_1+1)-2-\frac{1}{12}+\delta}$$

by (35) and (36), so the hypothesis corresponding to (3) of § 7 is satisfied. The argument is concluded as in the preceding case, but with reference now to the form C^+ in n_1+1 variables. We obtain the (conditional) asymptotic formula

$$\mathscr{N}(P) \sim P^{n_1-2} J^+_0 \mathfrak{S}^+$$

as $P \to \infty$, where $\mathfrak{S}^+$ is the singular series appropriate to C^+ and J^+_0 is the constant of lemma 6·2 appropriate to C^+. This completes the treatment of the present case.

(*d*) *The case* $[n_2; 2]$

We are now concerned with a cubic form in n_2+2 variables of the type

$$C(x_1, \ldots, x_{n_2})+d_1y_1^3+d_2y_2^3, \tag{37}$$

where $n_2 = n_1-6 \geqslant 18$. We regard this form alternatively as $C^+(x_1, \ldots, x_{n_2+2})$, and choose a corresponding box $\mathscr{B}^+$ in n_2+2 dimensions, which upon magnification gives a box $P\mathscr{B}$ in n_2 dimensions and two intervals for y_1 and y_2, each of the form (17). With a similar notation to that of the preceding case, we have

$$\mathscr{N}(P) = \int_I S(\alpha)\,T_1(\alpha)\,T_2(\alpha)\,\mathrm{d}\alpha, \tag{38}$$

and we dissect the integral as before.

We suppose that C does not split with remainder n_2-5, since this will be the next case. Thus we can take $\kappa = 3\theta(1-\delta)$ in lemma 4·2 and $\kappa = \frac{3}{2}-\delta$ in lemma 5·6. The conditions on n_2 in these lemmas are satisfied since $n_2 \geqslant 12$.

The work proceeds on the same lines as in the preceding case. The exponent corresponding to that in (20) is now $n_2-3\theta(1-\delta)$. In place of (22) we obtain

$$\int_{\mathscr{E}(\theta)} |\,T_1(\alpha)\,T_2(\alpha)\,|\,\mathrm{d}\alpha \ll P^{-\frac{1}{2}+2\theta+2\delta}. \tag{39}$$

The contribution of those α which satisfy the analogue of (20) when $\theta = \theta_1 = \frac{3}{4}$ is easily seen to be $\ll P^{n_2-1-\delta}$. The exponent corresponding to (24) is

$$< n_2-1+2(\theta_g-\theta_{g+1})-(\theta_{g+1}-\tfrac{1}{2})+5\delta,$$

and is less than $n_2-1-\delta$ under conditions similar to (25). There remains a set $\mathscr{F}$ in $\mathfrak{m}$ which is substantially the same as before. Using the analogue of (31) for $T_1(\alpha)$ and $T_2(\alpha)$, we find that

$$\int_{\mathscr{F}} |\,T_1(\alpha)\,T_2(\alpha)\,|\,\mathrm{d}\alpha \ll P^{\frac{1}{3}+\delta'}, \tag{40}$$

where δ' is a numerical multiple of δ. This suffices for the treatment of $\mathscr{F}$, since $|\,S(\alpha)\,| \ll P^{n_2-\frac{3}{2}+6\delta}$ throughout $\mathfrak{m}$, by the case $\theta = \frac{1}{2}(1-3\delta)$ of lemma 4·2. We obtain

$$\int_{\mathfrak{m}} |\,S(\alpha)\,T_1(\alpha)\,T_2(\alpha)\,|\,\mathrm{d}\alpha \ll P^{n_2-1-\delta}.$$

The treatment of the intervals $\mathfrak{M}_{a,q}$ is straightforward; we have now to multiply together an approximation like (33), but with $-\frac{1}{2}+2\delta$ for the exponent of q, and two approximations like (34). No difficulty arises, and we obtain finally a (conditional) asymptotic formula of the form

$$\mathscr{N}(P) \sim P^{n_2-1}J_0^+\,\mathfrak{S}^+$$

as $P \to \infty$. This completes the treatment of the case.

(*e*) *The case* $[n_3; 3]$

We are now concerned with a cubic form in n_3+3 variables of the type

$$C(x_1, \ldots, x_{n_3})+d_1y_1^3+d_2y_2^3+d_3y_3^3,$$

where $n_3 = n_2-5 \geqslant 13$. We suppose that C does not split with remainder n_3-4. We can take $\kappa = \frac{5}{2}\theta(1-\delta)$ in lemma 4·2 and $\kappa = \frac{5}{4}-\delta$ in lemma 5·6.

The work proceeds as before; in place of (22) we find that

$$\int_{\mathscr{E}(\theta)} |\, T_1(\alpha)\, T_2(\alpha)\, T_3(\alpha)\,|\, \mathrm{d}\alpha \ll P^{\frac{3}{4}+\theta+2\delta}.$$

The exponent corresponding to (24) is

$$< n_3 + (\theta_g - \theta_{g+1}) - \tfrac{3}{2}(\theta_{g+1} - \tfrac{1}{2}) + 5\delta,$$

and is less than $n_3 - \delta$ for a similar choice of $\theta_1, \theta_2, \ldots$. There remains a set $\mathscr{F}$ in $\mathfrak{m}$ as before. We find that

$$\int_{\mathscr{F}} |\, T_1(\alpha)\, T_2(\alpha)\, T_3(\alpha)\,|\, \mathrm{d}\alpha \ll P^{1+\delta'}, \tag{41}$$

and this suffices.

The treatment of the intervals $\mathfrak{M}_{a,q}$ is on the same lines as before, and we obtain a (conditional) asymptotic formula of the form

$$\mathscr{N}(P) \sim P^{n_3} J_0^+ \mathfrak{S}^+$$

as $P \to \infty$.

(*f*) *The case* $[n_4; 4]$

We are now concerned with a cubic form in $n_4 + 4$ variables of the type

$$C(x_1, \ldots, x_{n_4}) + d_1 y_1^3 + \ldots + d_4 y_4^3,$$

where $n_4 = n_3 - 4 \geqslant 9$. We suppose that C does not split with remainder $n_4 - 3$. We can take $\kappa = 2\theta(1-\delta)$ in lemma 4·2 and $\kappa = 1-\delta$ in lemma 5·6.

Although the general lines of proof are the same as before, a slight change of technique is appropriate in the treatment of $\mathfrak{m}$. In place of (22) we have

$$\int_{\mathscr{E}(\theta)} |\, T_1(\alpha) \ldots T_4(\alpha)\,|\, \mathrm{d}\alpha \ll P^{2+\epsilon}, \tag{42}$$

by immediate deduction from (21). Since

$$|\, S(\alpha)\,| \ll P^{n_4 - 2\theta(1-\delta)} \tag{43}$$

except in the set $\mathscr{E}(\theta)$, the use of several exponents θ is no longer necessary; for if integration is limited to those α which satisfy (43) when $\theta = \frac{1}{2} + 2\delta$, we have

$$\int |\, S(\alpha)\, T_1(\alpha) \ldots T_4(\alpha)\,|\, \mathrm{d}\alpha \ll P^{n_4+1-\delta}.$$

Thus there remains only a set of the same general type $\mathscr{F}$ as before. For the contribution of $\mathscr{F}$, we no longer use (31) directly but appeal to (41) and supplement this by the estimate

$$|\, T_4(\alpha)\,| \ll P^{\frac{3}{4}+\delta}. \tag{44}$$

This is Weyl's inequality, valid throughout $\mathfrak{m}$, and will be found as lemma 13 in Davenport (1939). Thus we obtain

$$\int_{\mathscr{F}} |\, T_1(\alpha) \ldots T_4(\alpha)\,|\, \mathrm{d}\alpha \ll P^{\frac{7}{4}+\delta'},$$

and since $|\, S(\alpha)\,| \ll P^{n_4-1+4\delta}$ in $\mathfrak{m}$, this suffices to complete the treatment of $\mathscr{F}$.

The contribution of the intervals $\mathfrak{M}_{a,q}$ is approximated to in the usual way, and we obtain an asymptotic formula in which the exponent of P is n_4+1.

(*g*) *The case* $[n_5;\, 5]$

We are now concerned with a cubic form in n_5+5 variables of the type

$$C(x_1, \ldots, x_{n5}) + d_1 y_1^3 + \ldots + d_5 y_5^3,$$

where $n_5 = n_4 - 3 \geqslant 6$. We can suppose that C does not split with remainder n_5-2. We can take $\kappa = \frac{3}{2}\theta(1-\delta)$ in lemma 4·2 and $\kappa = \frac{3}{4}-\delta$ in lemma 5·6. But it should be noted that the conditions imposed on n_5 by these choices are equivalent to $n_5 \geqslant 6$, and this condition is now only just satisfied.

In place of (22) we use

$$\int_{\mathscr{E}(\theta)} |\, T_1(\alpha) \ldots T_5(\alpha) \,|\, \mathrm{d}\alpha \ll P^{2+\frac{3}{4}+2\delta}, \tag{45}$$

which follows from (42) and (44). Again the use of several exponents θ is unnecessary. For the treatment of $\mathscr{F}$ we use

$$\int_{\mathscr{F}} |\, T_1(\alpha) \ldots T_5(\alpha) \,|\, \mathrm{d}\alpha \ll P^{2+\frac{1}{2}+\delta'}, \tag{46}$$

which follows from (41) and (44).

With these changes and the natural changes consequential on the new values of κ, the proof proceeds as in the last case.

(*h*) *The case* $[n_6;\, 6]$

We are now concerned with a cubic form in n_6+6 variables of the type

$$C(x_1, \ldots, x_{n6}) + d_1 y_1^3 + \ldots + d_6 y_6^3,$$

where $n_6 = n_5 - 2 \geqslant 4$. We suppose that C does not split with remainder n_6-1. We can take $\kappa = \theta(1-\delta)$ in lemma 4·2 and $\kappa = \frac{1}{2}-\delta$ in lemma 5·6. The conditions on n_6 become $n_6 \geqslant 4$, which is satisfied.

We now have

$$\int_{\mathscr{E}(\theta)} |\, T_1(\alpha) \ldots T_6(\alpha) \,|\, \mathrm{d}\alpha \ll P^{3+\frac{1}{2}+3\delta},$$

by (42) and (44). Also

$$\int_{\mathscr{F}} |\, T_1(\alpha) \ldots T_6(\alpha) \,|\, \mathrm{d}\alpha \ll P^{3+\frac{1}{4}+\delta'},$$

by (41) and (44). These results suffice for the treatment of $\mathfrak{m}$, and the treatment of the intervals $\mathfrak{M}_{a,q}$ presents no difficulty. It will be noted that the result of § 7 is still applicable, since C^+ is a form in $n_6+6 \geqslant 10$ variables.

(*i*) *The case* $[n_7;\, 7]$

We are now concerned with

$$C(x_1, \ldots, x_{n_7}) + d_1 y_1^3 + \ldots + d_7 y_7^3,$$

where $n_7 = n_6 - 1 \geqslant 3$. Putting $x_{n_1} = y_8$ and putting the other x's zero, we see that it suffices to prove that the form

$$d_1 y_1^3 + \ldots + d_8 y_8^3 \tag{47}$$

represents zero, where $d_1, \ldots, d_8$ are non-zero integers.

This is a fairly straightforward task, if one uses the methods and results of Davenport (1939). We first choose a small 7-dimensional box

$$\lambda_j < y_j < \mu_j \quad (j = 1, \ldots, 7), \tag{48}$$

which is centred on a real solution of $d_1 y_1^3 + \ldots + d_7 y_7^3 = 0$ with none of $y_1, \ldots, y_7$ zero. This defines 7 corresponding exponential sums $T_j(\alpha)$. We further define

$$W(\alpha) = \sum_{P^{4/5} < y_8 < 2P^{4/5}} \mathrm{e}\,(\alpha d_8 y_8^3). \tag{49}$$

The number of representations of zero by the form, with

$$\lambda_j P < y_j < \mu_j P \quad (1 \leqslant j \leqslant 7), \quad P^{4/5} < y_8 < 2P^{4/5} \tag{50}$$

is given by

$$\mathscr{N}(P) = \int_I T_1(\alpha) \ldots T_7(\alpha)\, W(\alpha)\, \mathrm{d}\alpha, \tag{51}$$

and the integral is again dissected as in (3).

It is easily deduced from lemma 1 of Davenport (1939) that

$$\int_0^1 |\, T_j(\alpha)\,|^2\, |\, W(\alpha)\,|^4\, \mathrm{d}\alpha \ll P^{\frac{13}{5}+\epsilon}$$

for any fixed $\epsilon > 0$. Further, by (21) and (44), we have

$$\int_{\mathfrak{m}} |\, T_j(\alpha)\,|^6\, \mathrm{d}\alpha \ll P^{2+\epsilon+\frac{3}{2}+2\delta}.$$

Hence, by Hölder's inequality,

$$\int_{\mathfrak{m}} |\, T_j(\alpha)\,|^5\, |\, W(\alpha)\,|\, \mathrm{d}\alpha \ll P^{\frac{1}{4}(\frac{13}{5}+\epsilon)+\frac{3}{4}(\frac{7}{2}+3\delta)} \ll P^{3+\frac{11}{40}+3\delta}. \tag{52}$$

It now follows from (52), (44) and Hölder's inequality that

$$\int_{\mathfrak{m}} |\, T_1(\alpha) \ldots T_7(\alpha)\, W(\alpha)\,|\, \mathrm{d}\alpha \ll P^{4+\frac{31}{40}+5\delta}.$$

This is a sufficiently good estimate for the contribution of $\mathfrak{m}$ to (51), for the main term in the final asymptotic formula will be of order $P^{-3}P^7P^{4/5} = P^{4+\frac{4}{5}}$.

It is necessary to deal separately with those $\mathfrak{M}_{a,q}$ for which $q > P^{\frac{4}{5}(1-\delta)}$, since effective approximation to $W(\alpha)$ is not possible in them. But their contribution can be estimated sufficiently well by taking a bound for $|\, T_j(\alpha)\,|$ $(j = 1, \ldots, 7)$ derived from its approximation and a bound for $|\, W(\alpha)\,|$ derived from (44), on the same lines as in Davenport (1939).

The intervals $\mathfrak{M}_{a,q}$ with $q \leqslant P^{\frac{4}{5}(1-\delta)}$ provide the main term in the asymptotic formula. Their contribution, apart from an error term which can be neglected, is

$$\sum_q \sum_a q^{-8}\, T_{ad_1,q} \ldots T_{ad_8,q} \int I^{(1)}(\beta) \ldots I^{(7)}(\beta)\, I^*(\beta)\, \mathrm{d}\beta, \tag{53}$$

where the integration is over $|\,\beta\,| < q^{-1}P^{-2-\delta}$, and where $I^{(1)}(\beta), \ldots, I^{(7)}(\beta)$ are as defined in (30), and where

$$I^*(\beta) = \int_{P^{4/5}}^{2P^{4/5}} \mathrm{e}\,(\beta d_8 \xi_8^3)\, \mathrm{d}\xi_8.$$

With $\xi_j = P\eta_j$ for $j \leqslant 7$ and $\xi_8 = P^{4/5}\eta_8$, and with $\beta = P^{-3}\phi$, the integral in (53) becomes

$$P^{4+\frac{4}{5}}\int_{-q^{-1}P^{1-\delta}}^{q^{-1}P^{1-\delta}}\left\{\int_{\lambda_1}^{\mu_1}\dots\int_{\lambda_7}^{\mu_7}\int_1^2 \mathrm{e}\,(\phi(d_1\eta_1^3+\dots+d_7\eta_7^3+P^{-3/5}d_8\eta_8^3))\,\mathrm{d}\eta_1\dots\mathrm{d}\eta_8\right\}\mathrm{d}\phi.$$

The effect, in (53), of ignoring the term $P^{-3/5}d_8\eta_8^3$ and the corresponding integration over η_8 is easily seen to be negligible. After this, the last expression becomes

$$P^{4+\frac{4}{5}}J(q^{-1}P^{1-\delta}),$$

where the function J is defined as in (3) of § 6, but relative to the form $d_1\eta_1^3+\dots+d_7\eta_7^3$ in the box $\lambda_j<\eta_j<\mu_j$ $(j=1,\dots,7)$. By the choice of this box made above, lemma 6·2 is applicable and gives

$$|\,J(q^{-1}P^{1-\delta})-J_0\,|<\epsilon(P)$$

for $q \leqslant P^{\frac{4}{5}(1-\delta)}$. Substituting in (53) and extending the series to infinity, the main term becomes

$$P^{4+\frac{4}{5}}J_0\sum_{q=1}^{\infty}\sum_{\substack{a=1\\(a,q)=1}}^{q} q^{-8}T_{ad_1,q}\dots T_{ad_8,q}.$$

The series here is the singular series for the form (47). It is absolutely convergent since

$$|\,T_{ad_j,q}\,|\ll q^{\frac{2}{3}},$$

as noted earlier. To know that its sum is positive, we need for every prime p the existence of a non-singular p-adic solution of

$$d_1y_1^3+\dots+d_8y_8^3=0,$$

a result analogous to that of § 2, but easier to prove because of the additive nature of the form. The proof in fact presents little difficulty, and the result with 7 terms instead of 8 has been proved by Lewis (1957 *a*).

We thus obtain the asymptotic formula

$$\mathcal{N}(P)\sim P^{4+\frac{4}{5}}J_0\mathfrak{S}$$

as $P\to\infty$, and in this case the result is unconditional. Hence the form (47) represents zero, and the proof of the theorem is complete.

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